3.4. **birth-death.** Continuous birth-death Markov chains are very similar to countable Markov chains. One new concept is "explosion" which means that an infinite number of state change events can happen in a finite amount of time.

3.4.1. *birth and explosion.* Suppose that people never die and that birth is a Poisson process.

$$X_t = \text{ population at time } t$$
$$S = \{1, 2, 3, \cdots\}$$
$$\lambda_n = \text{ rate of birth when population size is } n$$

In other words,

$$\alpha(n, n+1) = \lambda_n$$
$$\alpha(n, n) = -\lambda_n$$

and all other entries of the infinite matrix A are zero.

Question: What is the probability of *explosion*?

I.e., What is the probability that the population size will go to infinite in a finite amount of time?

Answer: This probability is either 0 or 1!

Proof. Suppose that

$$p = \mathbb{P}(\text{explosion occurs at some time } T < 1000)$$

Then either $p = 0$ or $p > 0$.

If $p = 0$ then the population will never explode.

If $p > 0$ then

$$\mathbb{P}(\text{no explosion in 1000 years}) = 1 - p < 1$$

If we wait N millennia then the probability of no explosion will be

$$(1-p)^N$$

Since this goes to 0 as $N \to \infty$, the probability of never having an explosion is zero. □

So, we just have to determine whether the population is destined to explode or not. To figure this out I computed the expected time it would take for the population size to go to infinity. If this time if finite we have an explosion.

$$T_1 = \text{ time it takes to jump } 1 \to 2$$
$$T_2 = \text{ time it takes to jump } 2 \to 3$$
$$T = \text{ time to explosion } = T_1 + T_2 + \cdots$$

$$\mathbb{E}(T) = \mathbb{E}\left(\sum T_n\right) = \sum \mathbb{E}(T_n) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

Theorem 3.10. *The population exploses a.s. if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

converges.

3.4.2. *conversion to discrete time.* Next I considered birth and death. But I didn't want 0 to be an absorbing state. So, I assumed that there was spontaneous generation.

$$X_t \in S = \{0, 1, 2, 3, \cdots\}$$

with all rates positive:

$$0 < \lambda_n = \text{ birth rate when } X_t = n, \quad n \geq 0$$

$$0 < \mu_n = \text{ death rate when } X_t = n, \quad n \geq 1$$

Let $J_1, J_2, \cdots$ be the jump times (the times when the population size jumps).

We convert to discrete time so that we can use the formulas that we already know:

$$Z_k = \text{ population size after } k \text{ birth-death events}$$

The transition probabilities for this countable Markov chain are given by:

Lemma 3.11.

$$\mathbb{P}(Z_{k+1} = n+1 \mid Z_k = n) = \frac{\lambda_n}{\lambda_n + \mu_n}$$

$$\mathbb{P}(Z_{k+1} = n-1 \mid Z_k = n) = \frac{\mu_n}{\lambda_n + \mu_n}$$

Proof. If we chop up the time into small intervals of length Δt then only one event will occur in each interval. The probability that there will be a birth in one of these time intervals is

$$\mathbb{P}(X_{t+\Delta t} = n+1 \mid X_t = n) \approx \lambda_n \Delta t$$

and the probability of death will be

$$\mathbb{P}(X_{t+\Delta t} = n-1 \mid X_t = n) \approx \mu_n \Delta t$$

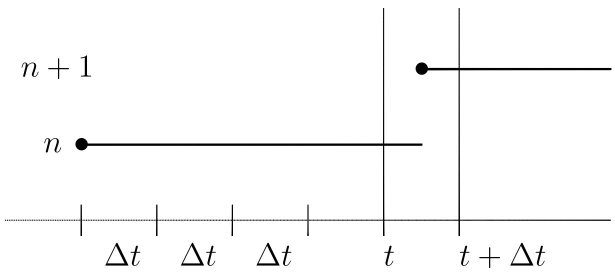
Since only one of these occur, the probability of a birth or death will be the sum:

$$\mathbb{P}(X_{t+\Delta t} \neq n \,|\, X_t = n) \approx \lambda_n \Delta t + \mu_n \Delta t$$

and the conditional probability of a birth will be:

$$\mathbb{P}(X_{t+\Delta t} = X_t + 1 \,|\, X_{t+\Delta t} \neq X_t) \approx \frac{\lambda_n \Delta t}{\lambda_n \Delta t + \mu_n \Delta t} = \frac{\lambda_n}{\lambda_n + \mu_n}$$

Taking the limit as $\Delta t \to 0$ we get:



$$\mathbb{P}(X_{t+dt} = X_t + 1 \,|\, X_{t+dt} \neq X_t) \approx \frac{\lambda_n dt}{\lambda_n dt + \mu_n dt} = \frac{\lambda_n}{\lambda_n + \mu_n}$$

□

(1) When is Z_n transient, null-recurrent, positive recurrent?
(2) When is X_t explosive?
(3) What is the relation between explosion and recurrence?

The correct answer to the last problem is that

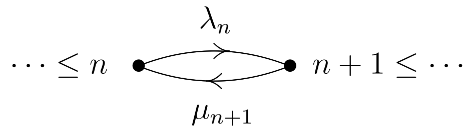
$$Z_n \text{ recurrent } \Rightarrow X_t \text{ nonexplosive}$$

The reason is that, when Z_n is recurrent, with probability one you return to the same state over and over. The expected return time is the same each time, say 100 years. If you return to the same place once every 100 years, it will take forever to return an infinite numbers of times. If you want to explode after that you don't have time.

Example 3.12. I asked these questions in the special case:

$$\lambda_n = \text{birth rate} = n + 1$$

$$\mu_n = \text{death rate} = n$$



3.4.3. *positive recurrent.* To see if X_t is positive recurrent we have to find an invariant distribution. This comes from the picture above. In order for the rate of movement from $\leq n$ to $\geq n+1$ to be the same as the reverse rate, the invariant distribution must satisfy:

$$\pi(n)\lambda_n = \pi(n+1)\mu_{n+1}$$

So,

$$\pi(n+1) = \pi(n)\frac{\lambda_n}{\mu_{n+1}} = \pi(n-1)\frac{\lambda_{n-1}\lambda_n}{\mu_n\mu_{n+1}}$$
$$= \cdots = \pi(0)\frac{\lambda_n\lambda_{n-1}\cdots\lambda_0}{\mu_{n+1}\mu_n\cdots\mu_1}$$

The sum of these numbers must be finite. So,

Theorem 3.13. *X_t is positive recurrent if and only if*

$$\sum_{n=0}^{\infty}\frac{\lambda_n\lambda_{n-1}\cdots\lambda_0}{\mu_{n+1}\mu_n\cdots\mu_1} < \infty$$

I.e., iff the series converges.

In the special case $\lambda_n = n+1, \mu_n = n$ each fraction is equal to 1. So, the sum is

$$1+1+1+1+\cdots = \infty$$

So, the series diverges and the Markov chain is not positive recurrent.

3.4.4. *transient.* To see if X_t is transient, we look at the discrete chain Z_n and try to find the function $\alpha(n)$ so that $\alpha(0) = 1$ and $\inf \alpha(n) = 0$. The equation for $\alpha(n)$ is:

$$\alpha(n) = \sum p(n,m)\alpha(m)$$

Since we can only go up and down by one this is:

$$\alpha(n) = p(n,n+1)\alpha(n+1) + p(n,n-1)\alpha(n-1)$$
$$\alpha(n) = \frac{\lambda_n\alpha(n+1)}{\lambda_n+\mu_n} + \frac{\mu_n\alpha(n-1)}{\lambda_n+\mu_n}$$
$$(\lambda_n+\mu_n)\alpha(n) = \lambda_n\alpha(n+1) + \mu_n\alpha(n-1)$$

This equation can be rewritten as

$$\mu_n(\underbrace{\alpha(n)-\alpha(n-1)}_{\Delta_n\alpha}) = \lambda_n(\underbrace{\alpha(n+1)-\alpha(n)}_{\Delta_{n+1}\alpha})$$
$$\mu_n\Delta_n\alpha = \lambda_n\Delta_{n+1}\alpha$$

So,

$$\Delta_{n+1}\alpha = \frac{\mu_n}{\lambda_n}\Delta_n\alpha = \frac{\mu_n}{\lambda_n}\frac{\mu_{n-1}}{\lambda_{n-1}}\Delta_{n-1}\alpha = \cdots$$
$$\alpha(n+1) - \alpha(n) = \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1}(\alpha(1) - \alpha(0))$$
$$\alpha(n) - \alpha(n-1) = \frac{\mu_{n-1} \cdots \mu_1}{\lambda_{n-1} \cdots \lambda_1}(\alpha(1) - \alpha(0))$$

Adding these up and canceling terms we get:

$$\alpha(n+1) - \alpha(0) = \sum_{k=0}^{n} \frac{\mu_{k-1} \cdots \mu_1}{\lambda_{k-1} \cdots \lambda_1}(\alpha(1) - \alpha(0))$$

Theorem 3.14. *X_t is transient if and only if*

$$\sum_{k=0}^{n} \frac{\mu_{k-1} \cdots \mu_1}{\lambda_{k-1} \cdots \lambda_1} < \infty$$

I.e., iff this series converges.

In the special case $\lambda_n = n+1, \mu_n = n$, the fraction is:

$$\frac{\mu_{k-1} \cdots \mu_1}{\lambda_{k-1} \cdots \lambda_1} = \frac{(k-1)\cdots 1}{k \cdots 2} = \frac{1}{k}$$

But this gives the harmonic series:

$$\sum \frac{1}{k} = \infty$$

So, the chains Z_n and X_t are not transient.

Therefore, it must be null-recurrent.

3.4.5. *explosion.* Recurrent implies nonexplosive. So, there is no explosion. The correct calculation which shows this is:

$$y_n = \text{ expected time it takes to get from } X = n \text{ to } X = n+1$$

$$y_n = \frac{1}{\lambda_n + \mu_n} + \frac{\mu_n}{\lambda_n + \mu_n}(y_{n-1} + y_n)$$
$$y_n = \frac{1}{2n+1} + \frac{n}{2n+1}(y_{n-1} + y_n)$$
$$\frac{n+1}{2n+1} y_n = \frac{1}{2n+1}\frac{n}{2n+1} y_{n-1}$$
$$y_n = 1 + \frac{n}{2n+1} y_{n-1} \geq 1$$

since $y_0 = 1$. So,

$$\sum y_n \geq 1 + 1 + 1 + \cdots = \infty$$

and there is no explosion.