4.5. **discounted payoff.** Here we assume that the payoff is losing value at a fixed rate so that after T steps it will only be worth $\alpha^T f(x)$ where α is the discount rate, say $\alpha = .90$. If there is no cost, the value function will satisfy the equation

$$v(x) = \max(f(x), \alpha(Pv)(x))$$

Again there is a recursive formula converging to this answer:

$$u_{n+1}(x) = \max(f(x), \alpha(Pu_n)(x))$$

where you start with

$$u_1(x) = \begin{cases} 0 & \text{if } x \text{ is absorbing} \\ f(x) & \text{if } f(x) \geq \alpha f(y) \text{ for all } y \\ \max \alpha f(y) & \text{otherwise} \end{cases}$$

Example 4.9. Random walk with absorbing walls:

$x =$	0	1	2	3	4	5
$f(x) =$	0	10	6	8	10	0
$\alpha =$	90%					

If $X_0 = 2$ then you get $f(x) = 6$ if you stop. But, if you continue you expect to get

$$\frac{f(x-1) + f(x+1)}{2} = \frac{10+8}{2} = 9$$

which will be worth

$$\alpha 9 = (.9)9 = 8.1 > 6$$

So, you should play even if there is a cost of $g(x) = 2$.

If $X_0 = 3$ then you get $f(x) = 8$ if you stop and

$$\alpha\left(\frac{f(x-1) + f(x+1)}{2}\right) = .9\left(\frac{6+10}{2}\right) = (.9)(8) = 7.2$$

if you continue. It looks like you should stop at $x = 3$. But we saw in class that this was not right.

Iteration algorithm: If you use $u_1 = (0, 10, 8.1, 7.2, 10, 0)$ then the iteration gives an sequence which increases and converges to $v(x)$ from below. That is because this vector is *subharmonic*. The superharmonic starting point is that $u_1(x)$ is either $f(x)$ or

$$\alpha \max f(y) = (.9)10 = 9$$

whichever is large. So

$$u_1(x) = (0, 10, 9, 9, 10, 0)$$

The vector Pu_1 is given by taking the average of the numbers on both sides:

Pu_1	αPu_1	$f(x)$	u_2
0	0	0	0
< 10	< 10	10	10
9.5	8.55	6	8.55
9.5	8.55	8	8.55
< 10	< 10	10	10
0	0	0	0

Pu_2	αPu_2	$f(x)$	u_3
0	0	0	0
< 10	< 10	10	10
9.275	8.3475	6	8.3475
9.275	8.3475	8	8.3475
< 10	< 10	10	10
0	0	0	0

Now you see that only the middle two coordinates of u change:

$$u_n = (0, 10, z, z, 10, 0)$$

The middle number converges:

$$\begin{array}{l} 8.3475 \\ 8.2564 \\ 8.2154\cdots \\ 8.1818 \end{array}$$

The important point is that it never goes below 8. So, the optimal strategy is to keep playing until you reach $x = 1$ or 4 and get $f(x) = 10$.

During the game you always have a 1/2 chance of stopping on the next turn. Your probability of stopping in two turns is 1/4, in 3 turns it is 1/8 and so on. Each time you play the final payoff decreases in value by a factor of $\alpha = .9$ So, your expected payoff at 2 or 3 is

$$\begin{aligned} v(2) = v(3) &= \frac{1}{2}\alpha \cdot 10 + \frac{1}{4}\alpha^2 \cdot 10 + \frac{1}{8}\alpha^3 \cdot 10 + \cdots \\ &= 10\left(\frac{\alpha}{2} + \left(\frac{\alpha}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^3 + \left(\frac{\alpha}{2}\right)^4 + \cdots\right) \\ &= \frac{\text{first term}}{1 - \text{ratio}} = \frac{5\alpha}{1 - \alpha/2} = \frac{10\alpha}{2 - \alpha} = \frac{9}{1.1} \\ &= 8\tfrac{2}{11} = 8\tfrac{18}{99} = 8.181818\cdots \end{aligned}$$

So,

$$v = (0, 10, 8\tfrac{2}{11}, 8\tfrac{2}{11}, 10, 0).$$