MATH 56A SPRING 2008
STOCHASTIC PROCESSES

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Contents

Date: April 2, 2008.

6. Renewal

A renewal process is an object or process which lasts for a certain amount of time which is random. When the object dies or the process stops then you replace the object with a new one or you restart the process from the beginning: You "renew" the process each time it stops and each process is independent of the previous ones (aside from the fact that it starts at the end of the previous process). I used the example of a light bulb. You put a light bulb into a socket and it lasts for a certain amount of time. When it burns out, we assume that it is immediately replaced by a new bulb. Each bulb is independent of the previous one.

Some numbers associated to this process are:

$$N_t := \text{ number of times the process is renewed in the time interval } (0, t]$$

Each complete process has a duration:

$$T_i := \text{ duration of the } i\text{th process}$$

If we start in the middle of a process then

$$Y := \text{ duration of process which is going on at time } 0$$

If we start with a renewal at time 0 then $Y = 0$.

I considered three kinds of light bulbs:

(1) The guaranteed light bulb which will last exactly 1000 hours.
(2) The Poisson light bulb. This light bulb is as good as new as long as it is working. Assume it has an expected life of 1000 hours. ($\lambda = 1/1000$).
(3) A light bulb which lasts:

$$T = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{2} \\ 1500\ hrs & \text{with probability } \frac{1}{2} \end{cases}$$

In all three cases,

$$\mu = \mathbb{E}(T) = 1000$$

where T is the length of time that the light bulb lasts.

The first question is: Which light bulb is worth more? The answer is that they are all worth the same. They all give an expected utility of 1000 hours of light. However, after many hours, something very interesting happens to the distribution of light bulbs.

The numbers that I spent a lot of time explaining are A_t, B_t, C_t:

$$A_t := \text{ the age of the current process at time } t$$

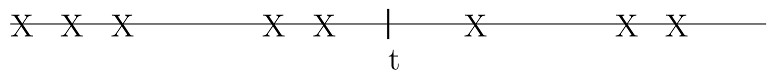
$$B_t := \text{ the remaining life of the current process at time } t$$

$$C_t = A_t + B_t = \text{ total life of the current process}$$

The question is: What are the equilibrium distributions of A_t, B_t, C_t ?

6.1. **relativity and equilibrium.** First, I used the example of the 500/1500 light bulb to explain the meaning of the equilibrium distribution. I also used “relativity” which says that, instead of thinking of random events happening in the future (at time $> t$) and the past (at times $\leq t$) we think of the entire timeline as given and the time t is a random point to be chosen on the timeline.

6.1.1. 500/1500 *light bulb.* For the 500/1500 light bulb, we have renewal events occurring on the timeline:



Half of the intervals of time (between the X's) are 500 hrs and half are 1500 hrs. If we pick a point t at random on this interval then the odds are 3:1 that it will lie on one of the 1500 hour intervals. So,

$$C_t = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{4} \\ 1500\ hrs & \text{with probability } \frac{3}{4} \end{cases}$$

If you walk into a warehouse which uses light bulbs of this kind then, at the beginning, when the light bulbs are new, half of them will be 500 hr light bulbs. After a year, a quarter of them will be 500 hr light bulbs. As time passes, the distribution and probabilities change. As an example, I asked the class to calculate the distribution of C_{1100}.

$$C_{1100} = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{8} \\ 1500\ hrs & \text{with probability } \frac{7}{8} \end{cases}$$

The reason is: There is only one way that the light bulb at time 1100 could be the short-life bulb. The first three bulbs must be of that kind. The probability of this is $(1/2)^3 = 1/8$.

General principle: At equilibrium, the longer life-spans become more likely.

Later, I gave a precise formulation and proved it.

As another example of relativity I asked the question: What is the probability that

$$B_t \geq A_t$$

The answer is 1/2 because, if we pick a time t at random, it will be equally likely that it is in the first half of a lifespan (gap between renewal events) as in the second half. The probability that it is exactly in the middle is 0.

6.1.2. *Poisson light bulb.* I did the example of the Poisson light bulb. Assume that we have a Poisson process with rate λ. We want to know the probability distribution of the remaining life B_t. This has probability density $f_B(t)$

$$f_B(s)ds = \mathbb{P}(s < B_t \leq s + ds)$$
$$= \mathbb{P}(\text{renewal in } (t+s, t+s+ds] \text{ and no renewal in } [t, t+s])$$
$$= (\lambda ds)(1 - \lambda ds)^{s/ds} = \lambda ds e^{-\lambda s}$$

So,

$$f_B(s) = \lambda e^{-\lambda s}$$

Since the argument and integral is just like what we did once or twice before with exactly the same pictures, I won't explain it here again.

6.2. **distribution of A_t, B_t, C_t.** On the second day I proved a bunch of theorems about the age and lifespan of the current process. I started with the following picture.

6.2.1. *picture for an expected value.*

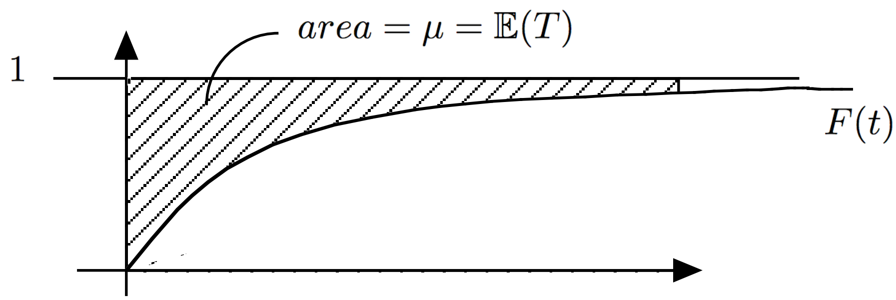


FIGURE 1. The shaded area above the distribution function $F(t)$ for T is equal to the expected value of T.

Theorem 6.1. *If $T \geq 0$ is a nonnegative random variable then the expected value of T is given by*

$$\mathbb{E}(T) = \int_0^\infty 1 - F(t)\,dt$$

I pointed out that t is a "dummy variable." So,

$$\int_0^\infty 1 - F(t)\,dt = \int_0^\infty 1 - F(s)\,ds$$

Proof. The expected value of T is defined by the integral

$$\mathbb{E}(T) := \int_0^\infty tF(t)\,dt$$

Substituting the integral

$$t = \int_0^t ds$$

we get:

$$\mathbb{E}(T) = \int_{t=0}^\infty \int_{s=0}^t ds\, f(t)\,dt$$

This is the integral of the density $f(t)$ over the region:

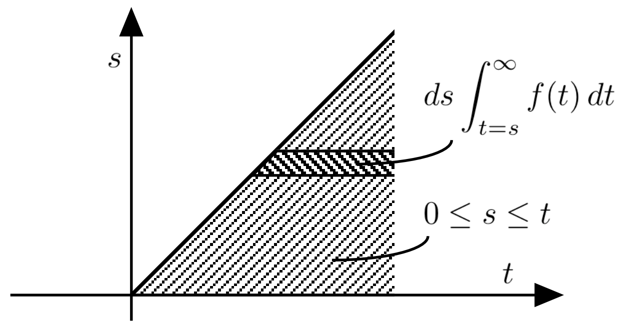


FIGURE 2. When this region is sliced up vertically, s runs from 0 to t. When it is sliced horizontally, t goes from s to ∞.

If we switch the order of integration we get:

$$\mathbb{E}(T) = \int_{s=0}^{\infty} \underbrace{\int_{t=s}^{\infty} f(t)\,dt}_{1-F(s)}\,ds$$

$$\mu = \mathbb{E}(T) = \int_0^{\infty} 1 - F(s)\,ds$$

□

Corollary 6.2.

$$\int_0^{\infty} \frac{1-F(t)}{\mu}\,dt = 1$$

6.2.2. *statement of the theorem.* We use the notation

$$T = \text{ life of one process}$$

with pdf $f(t)$, cdf $F(t)$ and expected value $\mu = \mathbb{E}(T)$.

Theorem 6.3. *The pdf of A_t, B_t, C_t are given in terms of $f(t) = f_T(t), F(t) = F_T(t)$ by:*

(1) $f_A(s) = \frac{1-F(s)}{\mu} = f_B(s)$
(2) $f_C(x) = \frac{xf(x)}{\mu}$

Note that the Corollary above says that $\frac{1-F(s)}{\mu}$ has integral 1 and is therefore a density function.

I used a relativity argument to explain why A_t, B_t have the same distribution. Namely, the entire process is *time reversible.* A renewal process is a sequence of time intervals of various durations which are independent of each other. The independence means we can reverse the order of events. If we reverse the entire timeline (run the film of events backwards) we would not see any difference. So, A_t, B_t are the same. C_t is different.

I used the example of the Poisson light bulb to illustrate the difference. The Poisson bulb is as good as new as long as it is working. So,

$$\mathbb{E}(B_t) = \mathbb{E}(T) = \frac{1}{\lambda} = \mu = 1000\ hrs$$

$$\mathbb{E}(A_t) = \frac{1}{\lambda} = \mu = 1000\ hrs$$

Since $\mathbb{E}$ is a linear function and $C_t = A_t + B_t$,

$$\mathbb{E}(C_t) = \mathbb{E}(A_t) + \mathbb{E}(B_t) = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda} = 2000\ hrs$$

Although each light bulb has an expected life of 1000 hours, the light bulbs currently in the sockets after one year will have total expected life of 2000 hours.

6.2.3. *renewal theorem.* In order to prove the theorem I needed another theorem:

Theorem 6.4 (Renewal Theorem)**.** *In equilibrium (as $t \to \infty$),*

$$\mathbb{P}(\text{ renewal in a time interval of length } \Delta s) = \boxed{\frac{\Delta s}{\mu}} + o(\Delta s)$$

Proof. If $\mu = \mathbb{E}(T)$ is 1000 then, in a span of t (a very large number), we expect the event to occur

$$\mathbb{E}(N_t) = \frac{t}{\mu} = \frac{t}{1000} \text{ times}$$

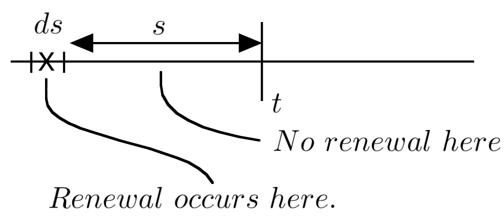
When t is very small, the expected number becomes less than 1 and it represents probability of occurrence.[1] So,

$$\mathbb{E}(N_{\Delta t}) = \frac{\Delta t}{\mu} \approx \mathbb{P}(\text{renewal in } (t, t + \Delta t])$$

□

[1] When the probability of the event happening more than once becomes negligible, expected number of occurrences becomes equal to probability of occurrence.

6.2.4. *distribution of A_t.*



We want to calculate $f_A(s)$.

$$f_A(s)ds = \mathbb{P}(s < A_t \leq s + ds)$$
$$= \mathbb{P}(\text{renewal occurs in } ds \text{ and it lasts longer than } s)$$

But,

$$\mathbb{P}(\text{renewal in } ds) = \frac{ds}{\mu}$$

by the renewal theorem and

$$\mathbb{P}(\text{life of process } > s) = 1 - F(s)$$

So,

$$f_A(s)ds = \frac{ds}{\mu}(1 - F(s))$$
$$f_A(s) = \frac{1 - F(s)}{\mu}$$

6.2.5. *distribution of C_t.* Next I proved:

$$f_C(x) = \frac{xf(x)}{\mu}.$$

Proof. Suppose that we wait for a very large number of renewals, say N. The total length of time that this takes will be about $N\mu$.

$$f(x)dx = \mathbb{P}(x < T \leq x + dx)$$

This represents the proportion of events whose duration is between x and $x + dx$. The number of events that we are talking about is:

$$Nf(x)dx = \text{ number of events which lasted } x \text{ to } x + dx \text{ amount of time}$$

Since each of these events lasted about x, the amount of time we are talking about is

$$xNf(x)dx = \text{ total duration of all these processes}$$

The proportion is the density function

$$f_C(x)dx = \text{proportion of the time spent in a process whose duration is between } x \text{ and } x+dx$$

$$= \frac{xNf(x)dx}{N\mu} = \frac{xf(x)dx}{\mu}$$

So,

$$f_C(x) = \frac{xf(x)}{\mu}$$

□

Example 6.5. For the Poisson process with rate λ,

$$f(t) = f_T(s) = \lambda e^{-\lambda t}, \qquad \mu = \frac{1}{\lambda}$$

$$F(t) = 1 - e^{-\lambda t}, \qquad 1 - F(t) = e^{-\lambda t}$$

$$f_A(s) = \frac{1 - F(s)}{\mu} = \lambda e^{-\lambda s} = f_T(s) = f_B(s)$$

So, the remaining life has the same distribution as the life of a new bulb. I.e., a used Poisson bulb is indistinguishable from a new one as long as it is working.

$$f_C(s) = \frac{sf(s)}{\mu} = s\lambda^2 e^{-\lambda s}$$

This is the *Gamma distribution* with parameters λ and $\alpha = 2$.

6.2.6. Γ-*distribution.*

Definition 6.6. The Γ-distribution with parameters λ, α is defined by the pdf

$$f_\Gamma(t) = \frac{1}{\Gamma(\alpha)}\lambda^\alpha t^{\alpha-1}e^{-\lambda t}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$$

This may be more familiar:

$$\Gamma(\alpha) = (\alpha - 1)!$$

So,

$$\Gamma(2) = 1! = 1$$

and the density function for $\Gamma(\lambda, 2)$ is

$$f(t) = \frac{1}{\Gamma(2)}\lambda^2 t^1 e^{-\lambda t} = \lambda^2 t e^{-\lambda t}$$

The intuitive definition is the following (when α is a positive integer). Let T_α be the time it takes for a Poisson event with rate λ to occur α times.

Theorem 6.7. *T_α is $\Gamma(\lambda, \alpha)$-distributed.*

Also, the *chi-square* distribution is an example of a Γ-distribution:

$$\chi^2_\nu = \Gamma\left(\frac{1}{2}, \frac{\nu}{2}\right)$$

In the case of the Poisson light bulb, A_t, B_t are independent random variables which are exponentially distributed. So,

$$C_t = A_t + B_t$$

must be $\Gamma(\lambda, 2)$-distributed by the theorem above since it is the sum of two waiting periods.

The next subtopic is convolution. This is the general formula for the density function of a random variable which is the sum of two independent random variables.

6.3. **convolution.**

Definition 6.8. Given two function f, g, their *convolution* $f * g$ is defined by

$$(f * g)(z) = \int_{-\infty}^{\infty} f(x) g(\underbrace{z - x}_{y})\, dx$$

This is often written as

$$(f * g)(z) = \int_{x+y=z} f(x) g(y)\, dx$$

The convolution is used to describe the density function for the sum of independent random variables. It occurs in this chapter because the lifespan of the renewal periods are independent. So, the density function for the n-th renewal is given by a convolution. I explained this in the particular example of the exponential distribution whose convolution gives the Γ-distribution.

6.3.1. *density of* $X + Y$.

Theorem 6.9. *Suppose that* X, Y *are independent random variables with density functions* $f_X(x), f_Y(y)$. *Then* $Z = X + Y$ *has density function:*

$$f_Z = f_X * f_Y$$

Proof. The definition of the density function is:

$$f_Z(z) dz = \mathbb{P}(z < Z \le z + dz) = \mathbb{P}(z < X + Y \le z + dz)$$

Since X, Y are independent, the joint density function is the product $f_X(x) f_Y(y)$. So, this is the integral:

$$\mathbb{P}(z < X + Y \le z + dz) = \int_{-\infty}^{\infty} \int_{z-x}^{z+x+dz} f_X(x) f_Y(y)\, dy dx$$

Figure 3 shows where the limits of integration came from. □

6.3.2. Γ-*distribution.*

Example 6.10. $f_X(x) = \lambda e^{-\lambda x}$ for $x \ge 0$ (exponential distribution) and $f_Y(y) = \lambda e^{-\lambda y}$ for $y \ge 0$. Then

$$f_X(x) f_Y(z - x) = \lambda e^{-\lambda z} \lambda e^{-\lambda(z-x)} = \lambda^2 e^{-\lambda z}$$

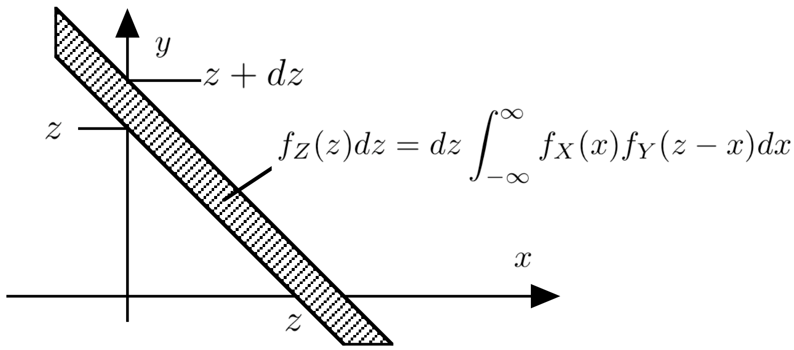


FIGURE 3. The mass of the strip is the probability that $z < Z \le z + dz$

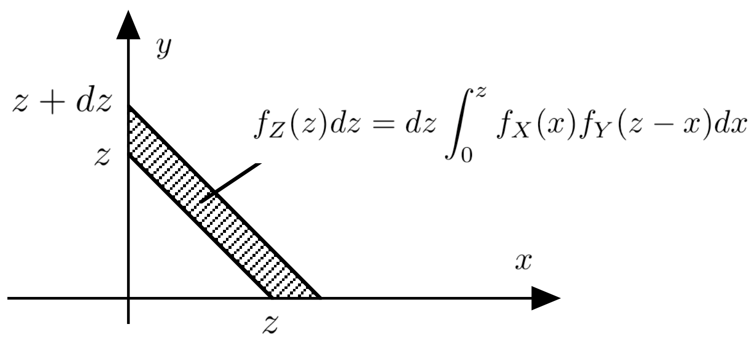


FIGURE 4. $x, y \ge 0$ means $0 \le x \le z$.

But this is true only if x, y are both ≥ 0. So, $z - x \ge 0$ or $z \ge x$. (See figure 4.) So,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = \int_0^z \lambda^2 e^{-\lambda z} dx = z\lambda^2 e^{-\lambda z}$$

(for $z \ge 0$). This is the Γ-distribution with $\alpha = 2$ and the given λ.

6.4. **M/G/1-queueing.** In this model, we have people lining up in a queue and one server taking care of these people one at a time. Let's assume the server is a machine.

In the notation "$M/G/1$" the "1" stands for the number of servers. The "M" means that the "customers" are entering the queue according to a Poisson process with some fixed rate λ. The "G" means that the servers does its job according to some fixed probability distribution which is "general." i.e., it could be anything. This is a renewal process where "renewal" occurs at the moment the queue is empty. At that time, the system is back in its original state with no memory of what happened.

$X_n = \#$ people who enter the line during the n-th service period.
$U_n =$ length of time to serve the n-th person.

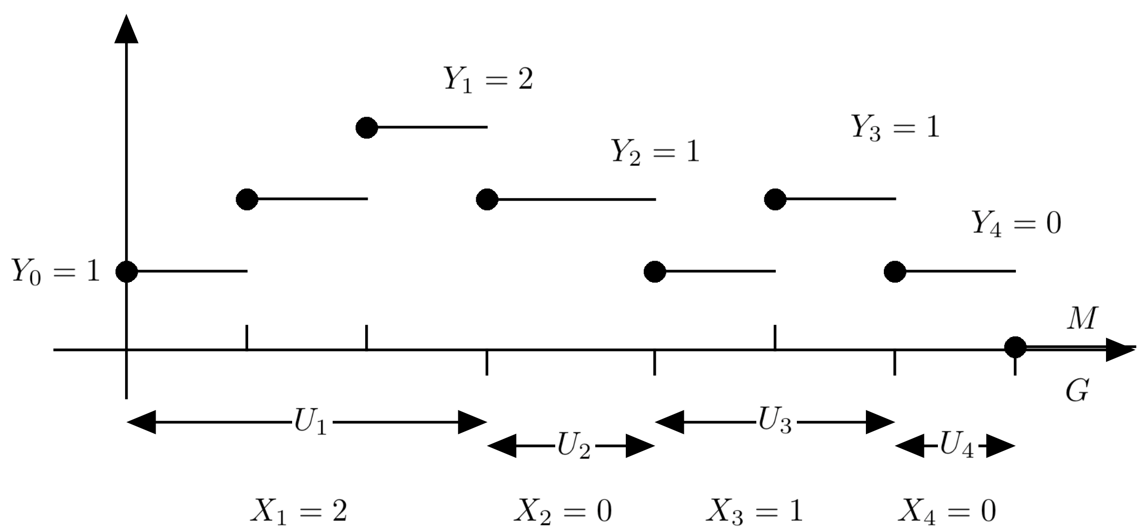
So, $\mathbb{E}(X_n) = \lambda\mu$ where $\mu = \mathbb{E}(U_n)$. We need to assume that $\lambda\mu < 1$. Otherwise, the line gets longer and long.

$Y_n = \#$ people in queue right after the n-th person has been served.

Then

$$Y_{n+1} - Y_n = X_{n+1} - 1$$

because X_{n+1} is the number of people who enter the line and one person leaves. (Let $Y_0 = 1$ so that the equation also holds for $n = 0$.)



6.4.1. *stopping time. Busy period* is when the queue and server are active. *Rest periods* are when there is noone in the line. The queue will alternate between busy periods and rest periods. The *stopping time* τ is the number of people served during the first busy period. Then the first busy time (duration of the 1st busy period) is

$$S_1 = U_1 + U_2 + \cdots + U_\tau$$

In the example drawn, $\tau = 4$ and $\sum X_i = 3 = \tau - 1$

To find a formula for τ we used exercise 5.16 on p.128:

(a) $M_n = X_1 + X_2 + \cdots + X_n - n\underbrace{\mathbb{E}(X)}_{\lambda\mu}$ is a uniformly integrable martingale.

(b) $M_0 = 0$

(c) $OST \Rightarrow \mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$. This gives us:

(d) (*Wald's equation*)

$$\mathbb{E}\left(X_1 + \cdots + X_\tau\right) = \mathbb{E}(\tau)\mathbb{E}(X)$$

But the sum of the numbers X_n gives the total number of people who entered the line after the first person. So:

$$X_1 + \cdots + X_\tau = \tau - 1$$

Put this into Wald's equation and we get:

$$\mathbb{E}(\tau) - 1 = \mathbb{E}(\tau)\mathbb{E}(X) = \mathbb{E}(\tau)\lambda\mu$$

where $\mu = \mathbb{E}(U)$. Solve for $\mathbb{E}(\tau)$ to get

$$\mathbb{E}(\tau) = \frac{1}{1 - \lambda\mu}$$

$$\mathbb{E}(S_1) = \mathbb{E}(\tau)\mu = \frac{\mu}{1 - \lambda\mu} = \frac{1}{\rho - \lambda}$$

where $\rho = 1/\mu$ is the service rate.

We want the expected value of $R_1 =$ 1st rest period

$$\mathbb{E}(R_1) = \frac{1}{\lambda}$$

because customers entering the queue is a Poisson event with rate λ.

$$\mathbb{E}(S_1 + R_1) = \mathbb{E}(S_1) + \mathbb{E}(R_1) = \frac{1}{\rho - \lambda} + \frac{1}{\lambda} = \frac{\lambda + \rho - \lambda}{\lambda(\rho - \lambda)} = \frac{\rho}{\lambda(\rho - \lambda)}$$

Therefore, the proportion of the time that the server if busy is:

$$\frac{\mathbb{E}(S)}{\mathbb{E}(S + R)} = \frac{1/(\rho - \lambda)}{\rho/\lambda(\rho - \lambda)} = \frac{\lambda}{\rho} = \lambda\mu < 1$$

Homework 6
Renewal

These problems are due Thursday, April 3. Answers will be posted the following week.

First problem: Suppose that we have a renewal process with a uniform distribution

$$f(t) = \begin{cases} 1/10 & \text{if } 0 < t \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(i.e., we have a light bulb which lasts at most 10 days and can burn out at any time with equal probability during those 10 days)

You walk into the warehouse after one year. The light bulbs are distributed according to the equilibrium distribution.

(1) What is the average age of a light bulb (in this equilibrium distribution)?
(2) If you pick a light bulb and it has been burning for 4 days, what is the probability that it will burn for at least 4 more days.

Second problem: a) Calculate the probability distribution of $Z = X + Y$ if X, Y are exponential variables with rate 2,3 resp.

b) Calculate the probability distribution of $W = X - Y$. [Hint: $-Y$ is a random variable with density function $f(y) = 3e^{3y}$ for $y < 0$ and $f(y) = 0$ if $y \geq 0$.

Third problem: We have a water tower which gives water to a desert town. It has been empty for a few day. :(

Suppose that it showers from time to time. This event is a Poisson process with rate $\lambda = 1/7$ (once a week on average). When it showers it always dumps 5,000 gallons of water into the tank. The residents of this town use 1,000 gallons of water per day continuously (at a constant rate throughout the day when there is water).

Convert this into an $M/G/1$ queue and determine:

(1) What is the meaning of X_n, Y_n, U_n, τ in this case?
(2) Compute μ.
(3) If there is a shower tomorrow, how long can the town expect to have water?