MATH 56A SPRING 2008
STOCHASTIC PROCESSES

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Contents

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6. Renewal

A renewal process is an object or process which lasts for a certain amount of time which is random. When the object dies or the process stops then you replace the object with a new one or you restart the process from the beginning: You "renew" the process each time it stops and each process is independent of the previous ones (aside from the fact that it starts at the end of the previous process). I used the example of a light bulb. You put a light bulb into a socket and it lasts for a certain amount of time. When it burns out, we assume that it is immediately replaced by a new bulb. Each bulb is independent of the previous one.

Some numbers associated to this process are:

$$N_t := \text{ number of times the process is renewed in the time interval } (0, t]$$

Each complete process has a duration:

$$T_i := \text{ duration of the } i\text{th process}$$

If we start in the middle of a process then

$$Y := \text{ duration of process which is going on at time } 0$$

If we start with a renewal at time 0 then $Y = 0$.

I considered three kinds of light bulbs:

(1) The guaranteed light bulb which will last exactly 1000 hours.
(2) The Poisson light bulb. This light bulb is as good as new as long as it is working. Assume it has an expected life of 1000 hours. ($\lambda = 1/1000$).
(3) A light bulb which lasts:

$$T = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{2} \\ 1500\ hrs & \text{with probability } \frac{1}{2} \end{cases}$$

In all three cases,

$$\mu = \mathbb{E}(T) = 1000$$

where T is the length of time that the light bulb lasts.

The first question is: Which light bulb is worth more? The answer is that they are all worth the same. They all give an expected utility of 1000 hours of light. However, after many hours, something very interesting happens to the distribution of light bulbs.

The numbers that I spent a lot of time explaining are A_t, B_t, C_t:

$$A_t := \text{ the age of the current process at time } t$$

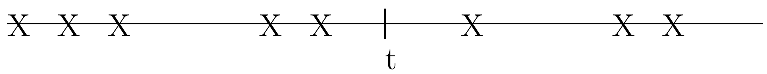
$$B_t := \text{ the remaining life of the current process at time } t$$

$$C_t = A_t + B_t = \text{ total life of the current process}$$

The question is: What are the equilibrium distributions of A_t, B_t, C_t ?

6.1. **relativity and equilibrium.** First, I used the example of the 500/1500 light bulb to explain the meaning of the equilibrium distribution. I also used "relativity" which says that, instead of thinking of random events happening in the future (at time $> t$) and the past (at times $\leq t$) we think of the entire timeline as given and the time t is a random point to be chosen on the timeline.

6.1.1. 500/1500 *light bulb.* For the 500/1500 light bulb, we have renewal events occurring on the timeline:



Half of the intervals of time (between the X's) are 500 hrs and half are 1500 hrs. If we pick a point t at random on this interval then the odds are 3:1 that it will lie on one of the 1500 hour intervals. So,

$$C_t = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{4} \\ 1500\ hrs & \text{with probability } \frac{3}{4} \end{cases}$$

If you walk into a warehouse which uses light bulbs of this kind then, at the beginning, when the light bulbs are new, half of them will be 500 hr light bulbs. After a year, a quarter of them will be 500 hr light bulbs. As time passes, the distribution and probabilities change. As an example, I asked the class to calculate the distribution of C_{1100}.

$$C_{1100} = \begin{cases} 500\ hrs & \text{with probability } \frac{1}{8} \\ 1500\ hrs & \text{with probability } \frac{7}{8} \end{cases}$$

The reason is: There is only one way that the light bulb at time 1100 could be the short-life bulb. The first three bulbs must be of that kind. The probability of this is $(1/2)^3 = 1/8$.

General principle: At equilibrium, the longer life-spans become more likely.

Later, I gave a precise formulation and proved it.

As another example of relativity I asked the question: What is the probability that

$$B_t \geq A_t$$

The answer is 1/2 because, if we pick a time t at random, it will be equally likely that it is in the first half of a lifespan (gap between renewal events) as in the second half. The probability that it is exactly in the middle is 0.

6.1.2. *Poisson light bulb.* I did the example of the Poisson light bulb. Assume that we have a Poisson process with rate λ. We want to know the probability distribution of the remaining life B_t. This has probability density $f_B(t)$

$$f_B(s)ds = \mathbb{P}(s < B_t \le s + ds)$$
$$= \mathbb{P}(\text{renewal in } (t+s, t+s+ds] \text{ and no renewal in } [t, t+s])$$
$$= (\lambda ds)(1 - \lambda ds)^{s/ds} = \lambda ds e^{-\lambda s}$$

So,

$$f_B(s) = \lambda e^{-\lambda s}$$

Since the argument and integral is just like what we did once or twice before with exactly the same pictures, I won't explain it here again.