

6.2. **distribution of A_t, B_t, C_t .** On the second day I proved a bunch of theorems about the age and lifespan of the current process. I started with the following picture.

6.2.1. *picture for an expected value.*

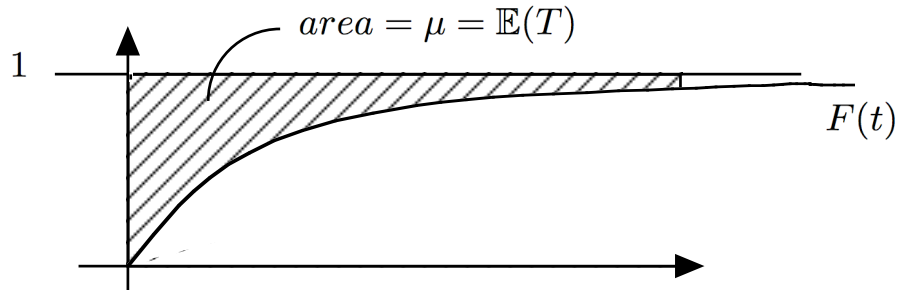


FIGURE 1. The shaded area above the distribution function $F(t)$ for T is equal to the expected value of T .

Theorem 6.1. *If $T \geq 0$ is a nonnegative random variable then the expected value of T is given by*

$$\mathbb{E}(T) = \int_0^{\infty} 1 - F(t) dt$$

I pointed out that t is a “dummy variable.” So,

$$\int_0^{\infty} 1 - F(t) dt = \int_0^{\infty} 1 - F(s) ds$$

Proof. The expected value of T is defined by the integral

$$\mathbb{E}(T) := \int_0^{\infty} tF(t) dt$$

Substituting the integral

$$t = \int_0^t ds$$

we get:

$$\mathbb{E}(T) = \int_{t=0}^{\infty} \int_{s=0}^t ds f(t) dt$$

This is the integral of the density $f(t)$ over the region:

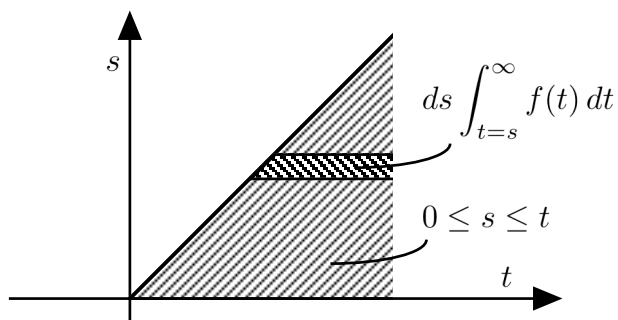


FIGURE 2. When this region is sliced up vertically, s runs from 0 to t . When it is sliced horizontally, t goes from s to ∞ .

If we switch the order of integration we get:

$$\mathbb{E}(T) = \int_{s=0}^{\infty} \underbrace{\int_{t=s}^{\infty} f(t) dt}_{1-F(s)} ds$$

$$\mu = \mathbb{E}(T) = \int_0^{\infty} 1 - F(s) ds$$

□

Corollary 6.2.

$$\int_0^{\infty} \frac{1 - F(t)}{\mu} dt = 1$$

6.2.2. *statement of the theorem.* We use the notation

$$T = \text{life of one process}$$

with pdf $f(t)$, cdf $F(t)$ and expected value $\mu = \mathbb{E}(T)$.

Theorem 6.3. *The pdf of A_t, B_t, C_t are given in terms of $f(t) = f_T(t), F(t) = F_T(t)$ by:*

- (1) $f_A(s) = \frac{1-F(s)}{\mu} = f_B(s)$
- (2) $f_C(x) = \frac{x f(x)}{\mu}$

Note that the Corollary above says that $\frac{1-F(s)}{\mu}$ has integral 1 and is therefore a density function.

I used a relativity argument to explain why A_t, B_t have the same distribution. Namely, the entire process is *time reversible*. A renewal process is a sequence of time intervals of various durations which are independent of each other. The independence means we can reverse the order of events. If we reverse the entire timeline (run the film of events backwards) we would not see any difference. So, A_t, B_t are the same. C_t is different.

I used the example of the Poisson light bulb to illustrate the difference. The Poisson bulb is as good as new as long as it is working. So,

$$\mathbb{E}(B_t) = \mathbb{E}(T) = \frac{1}{\lambda} = \mu = 1000 \text{ hrs}$$

$$\mathbb{E}(A_t) = \frac{1}{\lambda} = \mu = 1000 \text{ hrs}$$

Since \mathbb{E} is a linear function and $C_t = A_t + B_t$,

$$\mathbb{E}(C_t) = \mathbb{E}(A_t) + \mathbb{E}(B_t) = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda} = 2000 \text{ hrs}$$

Although each light bulb has an expected life of 1000 hours, the light bulbs currently in the sockets after one year will have total expected life of 2000 hours.

6.2.3. *renewal theorem*. In order to prove the theorem I needed another theorem:

Theorem 6.4 (Renewal Theorem). *In equilibrium (as $t \rightarrow \infty$),*

$$\mathbb{P}(\text{renewal in a time interval of length } \Delta s) = \boxed{\frac{\Delta s}{\mu}} + o(\Delta s)$$

Proof. If $\mu = \mathbb{E}(T)$ is 1000 then, in a span of t (a very large number), we expect the event to occur

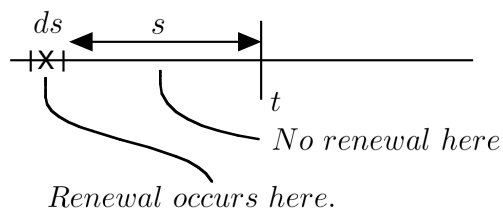
$$\mathbb{E}(N_t) = \frac{t}{\mu} = \frac{t}{1000} \text{ times}$$

When t is very small, the expected number becomes less than 1 and it represents probability of occurrence.¹ So,

$$\mathbb{E}(N_{\Delta t}) = \frac{\Delta t}{\mu} \approx \mathbb{P}(\text{renewal in } (t, t + \Delta t])$$

¹When the probability of the event happening more than once becomes negligible, expected number of occurrences becomes equal to probability of occurrence.

□

6.2.4. *distribution of A_t .*

We want to calculate $f_A(s)$.

$$\begin{aligned} f_A(s)ds &= \mathbb{P}(s < A_t \leq s + ds) \\ &= \mathbb{P}(\text{renewal occurs in } ds \text{ and it lasts longer than } s) \end{aligned}$$

But,

$$\mathbb{P}(\text{renewal in } ds) = \frac{ds}{\mu}$$

by the renewal theorem and

$$\mathbb{P}(\text{life of process} > s) = 1 - F(s)$$

So,

$$\begin{aligned} f_A(s)ds &= \frac{ds}{\mu}(1 - F(s)) \\ f_A(s) &= \frac{1 - F(s)}{\mu} \end{aligned}$$

6.2.5. *distribution of C_t .* Next I proved:

$$f_C(x) = \frac{xf(x)}{\mu}.$$

Proof. Suppose that we wait for a very large number of renewals, say N . The total length of time that this takes will be about $N\mu$.

$$f(x)dx = \mathbb{P}(x < T \leq x + dx)$$

This represents the proportion of events whose duration is between x and $x + dx$. The number of events that we are talking about is:

$Nf(x)dx =$ number of events which lasted x to $x + dx$ amount of time

Since each of these events lasted about x , the amount of time we are talking about is

$$xNf(x)dx = \text{total duration of all these processes}$$

The proportion is the density function

$$f_C(x)dx = \text{proportion of the time spent in a process whose duration is between } x \text{ and } x + dx$$

$$= \frac{xNf(x)dx}{N\mu} = \frac{xf(x)dx}{\mu}$$

So,

$$f_C(x) = \frac{xf(x)}{\mu}$$

□

Example 6.5. For the Poisson process with rate λ ,

$$f(t) = f_T(s) = \lambda e^{-\lambda t}, \quad \mu = \frac{1}{\lambda}$$

$$F(t) = 1 - e^{-\lambda t}, \quad 1 - F(t) = e^{-\lambda t}$$

$$f_A(s) = \frac{1 - F(s)}{\mu} = \lambda e^{-\lambda s} = f_T(s) = f_B(s)$$

So, the remaining life has the same distribution as the life of a new bulb. I.e., a used Poisson bulb is indistinguishable from a new one as long as it is working.

$$f_C(s) = \frac{sf(s)}{\mu} = s\lambda^2 e^{-\lambda s}$$

This is the *Gamma distribution* with parameters λ and $\alpha = 2$.

6.2.6. Γ -distribution.

Definition 6.6. The Γ -distribution with parameters λ, α is defined by the pdf

$$f_\Gamma(t) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha t^{\alpha-1} e^{-\lambda t}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

This may be more familiar:

$$\Gamma(\alpha) = (\alpha - 1)!$$

So,

$$\Gamma(2) = 1! = 1$$

and the density function for $\Gamma(\lambda, 2)$ is

$$f(t) = \frac{1}{\Gamma(2)} \lambda^2 t^1 e^{-\lambda t} = \lambda^2 t e^{-\lambda t}$$

The intuitive definition is the following (when α is a positive integer). Let T_α be the time it takes for a Poisson event with rate λ to occur α times.

Theorem 6.7. T_α is $\Gamma(\lambda, \alpha)$ -distributed.

Also, the *chi-square* distribution is an example of a Γ -distribution:

$$\chi_\nu^2 = \Gamma\left(\frac{1}{2}, \frac{\nu}{2}\right)$$

In the case of the Poisson light bulb, A_t, B_t are independent random variables which are exponentially distributed. So,

$$C_t = A_t + B_t$$

must be $\Gamma(\lambda, 2)$ -distributed by the theorem above since it is the sum of two waiting periods.

The next subtopic is convolution. This is the general formula for the density function of a random variable which is the sum of two independent random variables.