

### 6.3. convolution.

**Definition 6.8.** Given two function  $f, g$ , their *convolution*  $f * g$  is defined by

$$(f * g)(z) = \int_{-\infty}^{\infty} f(x) \underbrace{g(z-x)}_y dx$$

This is often written as

$$(f * g)(z) = \int_{x+y=z} f(x)g(y) dx$$

The convolution is used to describe the density function for the sum of independent random variables. It occurs in this chapter because the lifespan of the renewal periods are independent. So, the density function for the  $n$ -th renewal is given by a convolution. I explained this in the particular example of the exponential distribution whose convolution gives the  $\Gamma$ -distribution.

#### 6.3.1. density of $X + Y$ .

**Theorem 6.9.** Suppose that  $X, Y$  are independent random variables with density functions  $f_X(x), f_Y(y)$ . Then  $Z = X + Y$  has density function:

$$f_Z = f_X * f_Y$$

*Proof.* The definition of the density function is:

$$f_Z(z)dz = \mathbb{P}(z < Z \leq z + dz) = \mathbb{P}(z < X + Y \leq z + dz)$$

Since  $X, Y$  are independent, the joint density function is the product  $f_X(x)f_Y(y)$ . So, this is the integral:

$$\mathbb{P}(z < X + Y \leq z + dz) = \int_{-\infty}^{\infty} \int_{z-x}^{z+x+dz} f_X(x)f_Y(y) dydx$$

Figure 3 shows where the limits of integration came from. □

#### 6.3.2. $\Gamma$ -distribution.

**Example 6.10.**  $f_X(x) = \lambda e^{-\lambda x}$  for  $x \geq 0$  (exponential distribution) and  $f_Y(y) = \lambda e^{-\lambda y}$  for  $y \geq 0$ . Then

$$f_X(x)f_Y(z-x) = \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} = \lambda^2 e^{-\lambda z}$$

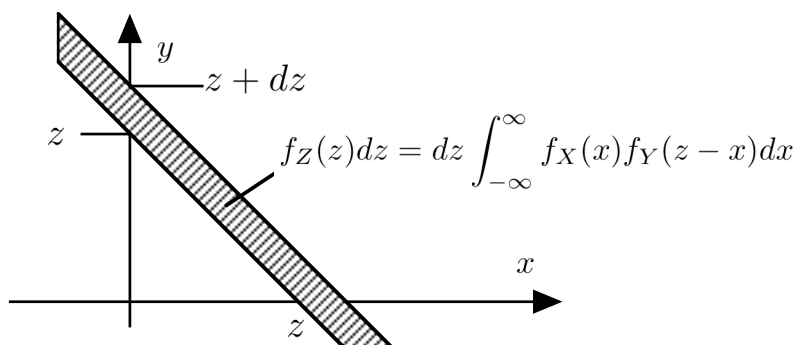


FIGURE 3. The mass of the strip is the probability that  $z < Z \leq z + dz$

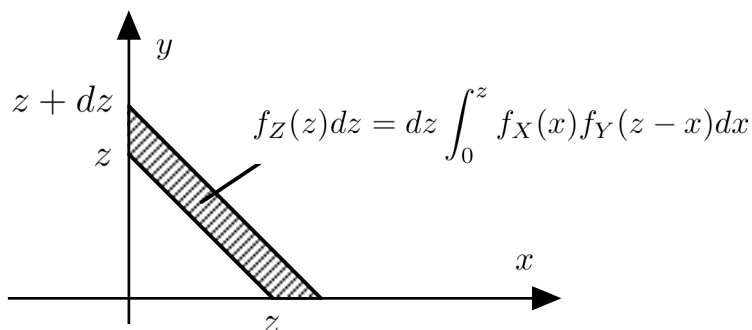


FIGURE 4.  $x, y \geq 0$  means  $0 \leq x \leq z$ .

But this is true only if  $x, y$  are both  $\geq 0$ . So,  $z - x \geq 0$  or  $z \geq x$ . (See figure 4.) So,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = \int_0^z \lambda^2 e^{-\lambda z} dx = z\lambda^2 e^{-\lambda z}$$

(for  $z \geq 0$ ). This is the  $\Gamma$ -distribution with  $\alpha = 2$  and the given  $\lambda$ .

**6.4. M/G/1-queueing.** In this model, we have people lining up in a queue and one server taking care of these people one at a time. Let's assume the server is a machine.

In the notation " $M/G/1$ " the "1" stands for the number of servers. The " $M$ " means that the "customers" are entering the queue according to a Poisson process with some fixed rate  $\lambda$ . The " $G$ " means that the servers does its job according to some fixed probability distribution

which is “general.” i.e., it could be anything. This is a renewal process where “renewal” occurs at the moment the queue is empty. At that time, the system is back in its original state with no memory of what happened.

$X_n = \#$  people who enter the line during the  $n$ -th service period.

$U_n =$  length of time to serve the  $n$ -th person.

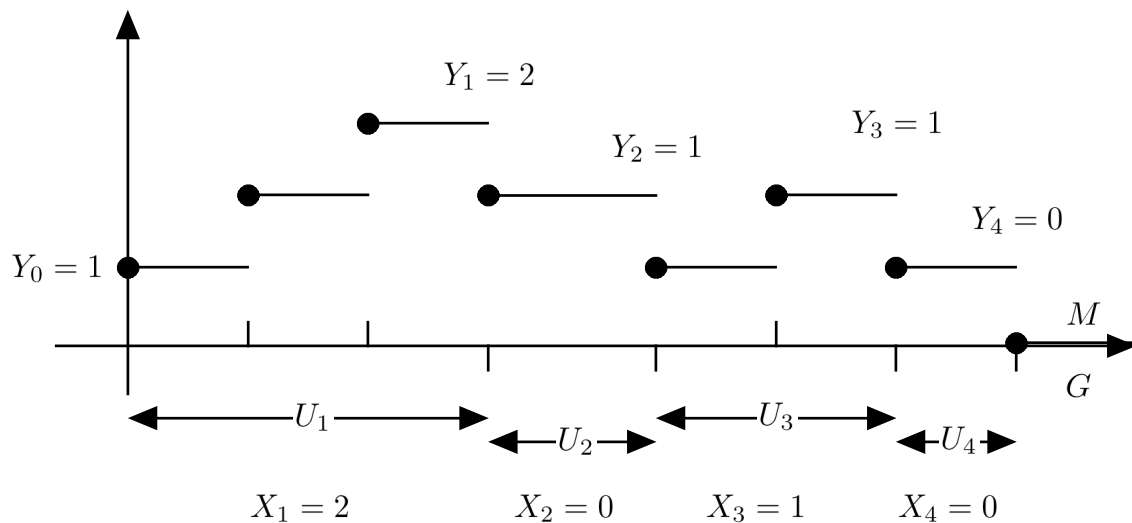
So,  $\mathbb{E}(X_n) = \lambda\mu$  where  $\mu = \mathbb{E}(U_n)$ . We need to assume that  $\lambda\mu < 1$ . Otherwise, the line gets longer and long.

$Y_n = \#$  people in queue right after the  $n$ -th person has been served.

Then

$$Y_{n+1} - Y_n = X_{n+1} - 1$$

because  $X_{n+1}$  is the number of people who enter the line and one person leaves. (Let  $Y_0 = 1$  so that the equation also holds for  $n = 0$ .)



6.4.1. *stopping time.* *Busy period* is when the queue and server are active. *Rest periods* are when there is no one in the line. The queue will alternate between busy periods and rest periods. The *stopping time*  $\tau$  is the number of people served during the first busy period. Then the first busy time (duration of the 1st busy period) is

$$S_1 = U_1 + U_2 + \dots + U_\tau$$

In the example drawn,  $\tau = 4$  and  $\sum X_i = 3 = \tau - 1$

To find a formula for  $\tau$  we used exercise 5.16 on p.128:

- (a)  $M_n = X_1 + X_2 + \cdots + X_n - n \underbrace{\mathbb{E}(X)}_{\lambda\mu}$  is a uniformly integrable martingale.  
 (b)  $M_0 = 0$   
 (c)  $OST \Rightarrow \mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$ . This gives us:  
 (d) (*Wald's equation*)

$$\mathbb{E}(X_1 + \cdots + X_\tau) = \mathbb{E}(\tau)\mathbb{E}(X)$$

But the sum of the numbers  $X_n$  gives the total number of people who entered the line after the first person. So:

$$X_1 + \cdots + X_\tau = \tau - 1$$

Put this into Wald's equation and we get:

$$\mathbb{E}(\tau) - 1 = \mathbb{E}(\tau)\mathbb{E}(X) = \mathbb{E}(\tau)\lambda\mu$$

where  $\mu = \mathbb{E}(U)$ . Solve for  $\mathbb{E}(\tau)$  to get

$$\mathbb{E}(\tau) = \frac{1}{1 - \lambda\mu}$$

$$\mathbb{E}(S_1) = \mathbb{E}(\tau)\mu = \frac{\mu}{1 - \lambda\mu} = \frac{1}{\rho - \lambda}$$

where  $\rho = 1/\mu$  is the service rate.

We want the expected value of  $R_1 =$  1st rest period

$$\mathbb{E}(R_1) = \frac{1}{\lambda}$$

because customers entering the queue is a Poisson event with rate  $\lambda$ .

$$\mathbb{E}(S_1 + R_1) = \mathbb{E}(S_1) + \mathbb{E}(R_1) = \frac{1}{\rho - \lambda} + \frac{1}{\lambda} = \frac{\lambda + \rho - \lambda}{\lambda(\rho - \lambda)} = \frac{\rho}{\lambda(\rho - \lambda)}$$

Therefore, the proportion of the time that the server is busy is:

$$\frac{\mathbb{E}(S)}{\mathbb{E}(S + R)} = \frac{1/(\rho - \lambda)}{\rho/\lambda(\rho - \lambda)} = \frac{\lambda}{\rho} = \lambda\mu < 1$$