8. Brownian Motion

We will spend 5 days on this chapter and cover the following topics:

8.1. introduction.

Brownian motion (named after Robert Brown) is also called a *Wiener process* (named after Norbert Wiener). It refers to the fact that particles bounce around under a microscope. They seem to be coming back to the same place but they actually come to a point just above or just below where they were before. You don't see the third dimension. (Random walk is recurrent in $\mathbb{Z}^2$ and transient in $\mathbb{Z}^3$.)

I started with general 1-dimensional Brownian motion. This means we are looking just at the x-coordinate of a particle moving in 3-dimensional space. You need to imagine that the particle is moving at random inside of a medium which is drifting.

X_t = position of a particle on a line at time t. This is the x-coordinate of a particle in $\mathbb{R}^3$. The concept is verbally described as follows.

(1) *memoryless* If $s > t$ then X_s depends only on X_t and not on how the particle got to the point X_t. More succinctly, $X_s - X_t$ is independent of $\mathcal{F}_t$ (all X_u for $u \leq t$).
(2) *time & space homogeneous* The increment in position depends only on the increment in time. I.e., the probability distribution of $X_{t+\Delta t} - X_t$ depends only on Δt. (It is independent of t, X_t.)
(3) X_t is *continuous*.

8.1.1. *mathematical definition.*

Definition 8.1. A random variable $X : [0, \infty) \to \mathbb{R}$ ($t \mapsto X_t$) is *Brownian motion* with *drift* μ and *variance* σ^2 (the book calls this the *variance parameter*) if

(1) For all $s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n$ [This means $(s_1, t_1], (s_2, t_2], \cdots, (s_n, t_n]$ are disjoint, i.e., nonoverlapping.]

$$X_{t_1} - X_{s_1}, X_{t_2} - X_{s_2}, \cdots, X_{t_n} - X_{s_n}$$

are independent random variables.
(2) X_t is continuous (it doesn't jump)

(3) $X_t - X_s$ is normally distributed with mean $(t-s)\mu$ and variance $|t-s|\sigma^2$. This is $(t-s)\sigma^2$ assuming $t > s$. So:

$$\boxed{X_t - X_s \sim N(\mu(t-s), \sigma^2|t-s|)}$$

$\sim$ means "distributed as".

We can convert this to the standard normal distribution by subtracting the mean and dividing by the standard deviation:

$$\frac{X_t - X_s - \mu(t-s)}{\sigma\sqrt{|t-s|}} \sim N(0,1)$$

Definition 8.2. *Standard Brownian motion*, denoted W_t, is Brownian motion with three additional conditions:

a) $W_0 = 0$ (W_t is "centered")
b) $\mu = 0$ (no drift)
c) $\sigma^2 = 1$.

For example, if X_t is Brownian motion with drift μ and variance σ^2 then

$$W_T = \frac{X_t - X_0 - \mu t}{\sigma}$$

is standard Brownian. Solving this for X_t we get:

$$\boxed{X_t = X_0 + \mu t + \boxed{\sigma W_t}}$$

The book assumes that $X_0 = 0$ and $\mu = 0$. This focuses on the *stochastic part* of X_t which is σW_t. (X_0 is $\mathcal{F}_0$-measurable and μt is not random.)

Theorem 8.3. *Condition (3) in the definition of Brownian motion is equivalent to the condition that* $X_{t_1} - X_{s_1}, X_{t_2} - X_{s_2}, \cdots$ *are i.i.d. with finite mean and variance for nonoverlapping intervals* $(s_1, t_1], (s_2, t_2], \cdots$ *of the same length:*

$$t_1 - s_1 = t_2 - s_2 = \cdots$$

I pointed out that this condition follows from the assumptions of being "memoryless" and "time & space homogeneous." In other words, this theorem says that the verbal and mathematical descriptions of Brownian motion agree.

Proof. Choose Δt small: $\Delta t = \frac{t-s}{N}$. Then

$$X_t - X_s = (X_{s+\Delta t} - X_s) + (X_{s+2\Delta t} - X_{s+\Delta t}) + \cdots + (X_t - X_{t-\Delta t})$$

The RHS is a sum of N i.i.d. random variables and we can take $N \to \infty$ without changing the LHS. Therefore, by the central limit theorem,

$X_t - X_s$ is normally distributed. We just need to compute its mean and standard deviation. But:

$$\mathbb{E}(X_t - X_s) = \underbrace{N}_{\frac{t-s}{\Delta t}} \mathbb{E}(X_{s+\Delta t} - X_s)$$

$$\mathbb{E}(X_t - X_s) = (t-s)\frac{\mathbb{E}(X_{s+\Delta t} - X_s)}{\Delta t} = (t-s)\mu$$

where μ is defined by

$$\mu := \frac{\mathbb{E}(X_{s+\Delta t} - X_s)}{\Delta t}$$

This does not depend of s, t or Δt since

$$\frac{\mathbb{E}(X_t - X_s)}{t-s} = \frac{\mathbb{E}(X_{s+\Delta t} - X_s)}{\Delta t}$$

the LHS does not depend on Δt and the RHS does not depend on s or t.

A similar trick works for the variance:

$$\mathrm{Var}(X_t - X_s) = N\,\mathrm{Var}(X_{s+\Delta t} - X_s)$$

$$\mathrm{Var}(X_t - X_s) = (t-s)\frac{\mathrm{Var}(X_{s+\Delta t} - X_s)}{\Delta t} = (t-s)\sigma^2$$

where σ^2 is defined by

$$\sigma^2 := \frac{\mathrm{Var}(X_{s+\Delta t} - X_s)}{\Delta t}$$

This works because, again, there is no t in this expression.

Putting these together we get

$$X_t - X_s \sim N((t-s)\mu, (t-s)\sigma^2)$$

assuming $t > s$. $\square$

8.1.2. *as a limit of random walks.* Standard Brownian motion is a limit (as $\Delta t \to 0$) of random walks where:

<u>Time</u> moves in integer multiples of Δt and

<u>Space</u> is an integer multiple of $\sqrt{\Delta t}$

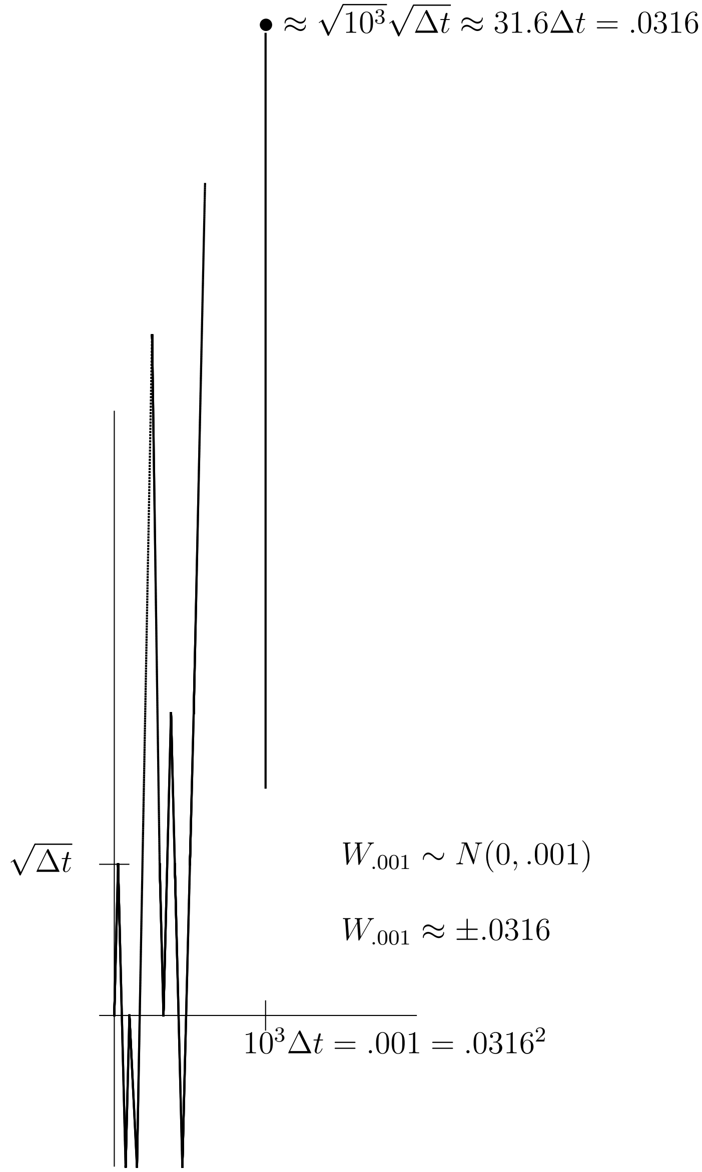
The <u>Markov chain</u> is: In each increment Δt of time, you move $\pm\sqrt{\Delta t}$ with probability 1/2.

Note that $\Delta t << \sqrt{\Delta t}$. For example,

$$\Delta t = 10^{-6} = \frac{1}{1,000,000} \Rightarrow \sqrt{\Delta t} = 10^{-3} = \frac{1}{1,000} = 10^3 \Delta t$$

The cover of our book shows a graph of a typical motion of this Markov chain. But, since Δt is going to 0, it would be more accurate to use

$\Delta t = 10^{-6}$. Then, $\sqrt{\Delta t} = 10^3 \Delta t$. So, a more accurate picture would be:



Since $X_{\Delta t} = \pm\sqrt{\Delta t}$ has mean 0 and variance Δt, the CLT says that

$$X_{N\Delta t} \dot{\sim} N(0, N\Delta t)$$

As $\Delta t \to 0$ (with $t = N\Delta t$ fixed), X_t converges to

$$W_t = X_t \sim N(0, t)$$

8.1.3. *Lévy's theorem.* I stated the following theorem without proof because I ran out of time. But the proof is easy.

Theorem 8.4. *Suppose that X_t is Brownian motion with drift μ and variance σ^2. Then*

a) *$X_t - \mu t$ is a continuous martingale.*
b) *$(X_t - \mu t)^2 - t\sigma^2$ is a continuous martingale.*

Proof. (a) Let $M_t = X_t - \mu t$. Then:

$$\mathbb{E}(M_t \,|\, \mathcal{F}_s) = \mathbb{E}(X_t - \mu t \,|\, \mathcal{F}_s) = X_s - \mu t + \mathbb{E}(X_t - X_s)$$
$$= X_t - \mu t + (t-s)\mu = X_s - \mu s = M_s$$

(b) $M_t = X_t - \mu t$ is Brownian motion with zero drift. Therefore, $\mathbb{E}(M_t - M_s) = 0$ and $\mathbb{E}((M_t - M_s)^2) = \sigma^2(t-s)$ for $t > s$. So,

$$\mathbb{E}(M_t^2 - \sigma^2 t \,|\, \mathcal{F}_s) = M_s^2 - \sigma^2 t + \mathbb{E}(M_t^2 - M_s^2 \,|\, \mathcal{F}_s)$$

But,

$$M_t^2 - M_s^2 = (M_t - M_s)^2 + 2M_2(M_t - M_s)$$

with expected value

$$\mathbb{E}(M_t^2 - M_s^2 \,|\, \mathcal{F}_s) = \mathbb{E}((M_t - M_s)^2 \,|\, \mathcal{F}_s) + 2M_s \underbrace{\mathbb{E}(M_t - M_s \,|\, \mathcal{F}_s)}_{0}$$
$$= \sigma^2(t-s)$$

So,

$$\mathbb{E}(M_t^2 - \sigma^2 t \,|\, \mathcal{F}_s) = M_s^2 - \sigma^2 t + \sigma^2(t-s) = M_s^2 - \sigma^2 s$$

making $M_t^2 - \sigma^2 t$ into a martingale. □

So, you can see that this is easy. Lévy proved the difficult converse of this theorem. One of the most elegant proofs of this theorem by Kunita and Watanabe is outlined in section 9.5 of the 2006 notes.

Theorem 8.5 (Lévy)**.** *A continuous L^2 martingale M_t is standard Brownian motion if and only if $M_0 = 0$ and $M_t^2 - t$ is a martingale.*