8.2. **strong Markov property and reflection principle.** These are concepts that you can use to compute probabilities for Brownian motion.

8.2.1. *strong Markov property.* a) Brownian motion satisfies the *Markov property*: For $t > s$, "X_t depends only on X_s" the book tries to write this mathematically. In class we decided that the statement is "$\mathcal{F}_s$ is given by X_s"

In fact (a fortiori), $X_t - X_s$ is independent of $\mathcal{F}_s$.

b) The *strong Markov property* says the same thing with s replaced by a stopping time T: If T is a stopping time and $t > T$ then "X_t depends only on $\mathcal{F}_T$"

In fact, we have the following theorem:

Theorem 8.6. *Suppose that X_t is Brownian motion. If $t > T$, T a stopping time, then $X_t - X_T$ is independent of $\mathcal{F}_T$.*

An example of a stopping time is the first time that X_t reaches 1.

8.2.2. *reflection principle.* Suppose X_t is Brownian motion with zero drift ($\mu = 0$). Then we want to calculate the probability that, starting at $X_0 = 0$, it will reach $X_s = 1$ at some time $0 < s < t$.

$$\mathbb{P}(X_s = 1 \text{ for some } 0 < s < t \,|\, X_0 = 0) = ?$$

Let T = first time that $X_T = 1$. Then X_s reaches 1 for $s < t$ if $T < t$. So, this is the same as

$$\mathbb{P}(T < t)$$

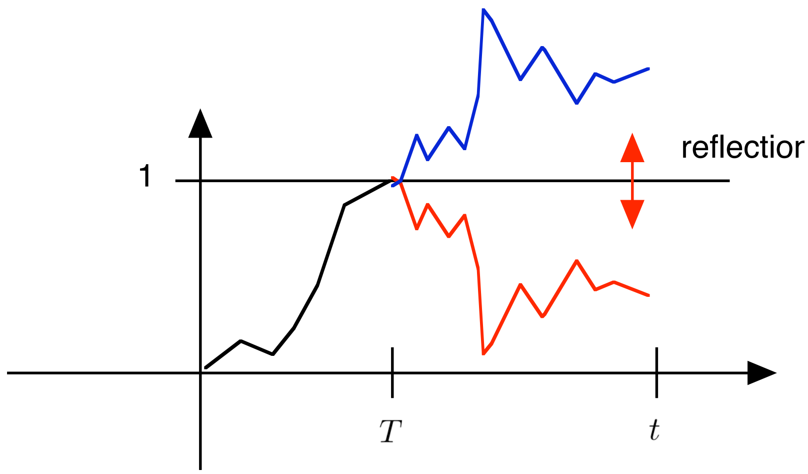
The strong Markov property implies that $X_t - X_T$ is independent of $\mathcal{F}_T$. We also know that $X_t - X_T$ is normal:

$$X_t - X_T \sim N(0, \sigma^2(t - T))$$

(assuming that $t > T$). Since the mean is zero, it is positive half the time and negative half the time (and the probability of being exactly zero is 0):

$$\mathbb{P}(X_t - X_T > 0) = \frac{1}{2}$$

$$\mathbb{P}(X_t - X_T \leq 0) = \frac{1}{2}$$



Half the time X will reach 1 and go up, half the time it will reach 1 and go down. So,

$$\mathbb{P}(T < t) = 2\mathbb{P}(T < t \text{ and } X_t > X_T = 1)$$

But X_t is continuous. So, the intermediate value theorem (IMT) tells us that the second condition implies the first: If $X_t > 1$ and $X_0 = 0$ then $0 < \exists s < 1$ so that $X_s = 1$. So,

$$\mathbb{P}(T < t \,|\, X_0 = 0) = 2\mathbb{P}(X_t > 1 \,|\, X_0 = 0)$$

This is given by an integral

$$= 2\int_1^\infty f_t(x)\, dx$$

where f_t is the density function for $X_t - X_0$.

8.2.3. *density function for normal distribution.* Since

$$X_t - X_0 \sim N(0, \sigma^2 t)$$

the density function is

$$f_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-x^2/2\sigma^2 t}$$

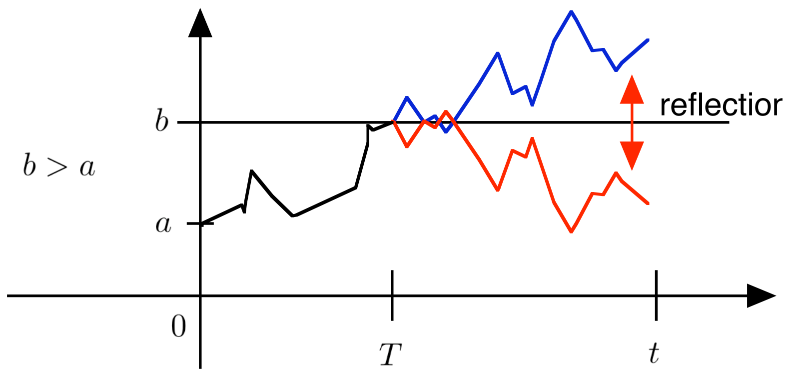
In other words,

$$f_t(x)ds = \mathbb{P}(x < X_t \leq x + dx \,|\, X_0 = 0)$$

Combine this with the refection principle gives:

$$\mathbb{P}(T < t) = 2\int_1^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-x^2/2\sigma^2 t}\, dt$$

To compute this (or look it up in a table) we should convert to standard normal. But first, I redid the reflection principle with 0,1 replace with a, b:



$$\begin{aligned} &\mathbb{P}(X_s = b \text{ for some } 0 < s < t \,|\, X_0 = a) \\ &= 2\mathbb{P}(X_t > b \,|\, X_0 = a) \qquad \text{by the reflection principle} \\ &= 2\int_b^\infty f_t(x-a)\,dx \end{aligned}$$

since $X_t - X_0 = X_t - a \sim N(0, \sigma^2 t)$ this integral is

$$= 2\int_b^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(x-a)^2/2\sigma^2 t}\,dx$$

8.2.4. *conversion to standard normal.* To convert to standard normal ($N(0,1)$) we should subtract the mean and divide by the standard deviation:

$$y = \frac{x-a}{\sigma\sqrt{t}}, \qquad dy = \frac{dx}{\sigma\sqrt{t}}$$

This converts the integral into:

$$\mathbb{P}(X_s = b \text{ for some } 0 < s < t \,|\, X_0 = a) = 2\int_{\frac{b-a}{\sigma\sqrt{t}}}^\infty \underbrace{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}_{\phi_1(y)}\,dy$$

I will use the abbreviation:

$$\phi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

This is the density function for $X_t - X_0$ if X_t is standard Brownian motion.

8.2.5. *Chapmann-Kolmogorov equation.* This is an "obvious" equation which I proved using the theorem that the density function of a sum of two random variables is the convolution of the density functions. First some notation:

$$p_t(x, y) = \text{ probability density of going from } x \text{ to } y \text{ in time } t$$

Multiply dy to get an actual probability:

$$p_t(x, y)dy = \mathbb{P}(y < X_{s+t} \leq y + dy \,|\, X_s = x)$$

So,

$$p_t(x, y) = f_t(y - x).$$

Theorem 8.7 (Chapmann-Kolmogorov)**.**

$$\boxed{p_{s+t}(x, y) = \int_{-\infty}^{\infty} p_s(x, z)p_t(z, y)\, dz}$$

Proof. Since $p_{s+t}(x, y) = f_{s+t}(y - x)$ we can rewrite this as:

$$f_{s+t}(y - x) = \int_{-\infty}^{\infty} f_s(z - x)f_t(y - z)dz$$

But $(z - x) + (y - z) = y - x$. So, the RHS is the convolution of f_s and f_t. But,

$$X_{s+t} - X_0 = (X_s - X_0) + (X_{s+t} - X_s)$$

So,

$$\begin{pmatrix}\text{density of}\\ X_{s+t} - X_0\end{pmatrix} = \begin{pmatrix}\text{density of}\\ X_s - X_0\end{pmatrix} * \begin{pmatrix}\text{density of}\\ X_{s+t} - X_s\end{pmatrix}$$

I.e.,

$$f_{s+t} = f_s * f_t$$

So, LHS=RHS. $\square$

The reason that this is supposed to be obvious is that, in order to go from x to y in time $s + t$ you have to first go to some z at time s and then get from z to y in the remaining time t. Since z could be anything you integrate over all z. This integral is the continuous version of matrix multiplication.

8.2.6. *return probability.* We used Chapmann-Kolmogorov to compute return probability. For this I assumed that $X_t = W_t$ is standard Brownian. So, $\mu = 0, \sigma = 1, X_0 = 0$. We want the return probability:

$$\mathbb{P}(X_s = 0 \text{ for some } 1 < s < t \,|\, X_0 = 0)$$

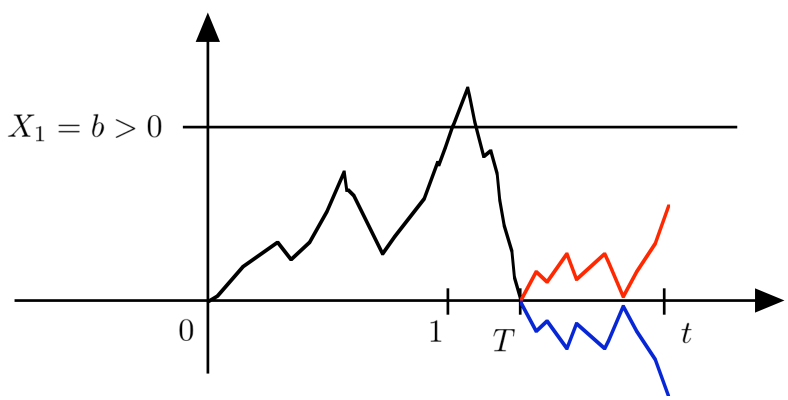
Later we will replace 1 with an arbitrary number.

For this problem we use the reflection principle twice: Half the time X_1 will be positive:

$$\mathbb{P}(X_1 > 0) = \frac{1}{2}$$

And, given that $X_1 = b > 0$ and $X_s = 0$ then half the time $X_t < 0$. So, by the reflection principle:

$$\mathbb{P}(X_s = 0 \text{ for some } 1 < s < t \,|\, X_0 = 0) = 4\mathbb{P}(X_1 > 0 \text{ and } X_t < 0)$$



By Chapmann-Kolmogorov (with $x, y, z = X_0, X_t, X_1$ resp.) this is

$$= 4\int_0^\infty \mathbb{P}(\underbrace{b < X_1 \le b + db}_{\phi_1(b)db} \text{ and } \underbrace{X_t - X_1 < -b}_{(*)})$$

But these two conditions are independent. So, we multiply the probabilities:

$$\mathbb{P}(b < X_1 \le b + db) = \phi_1(b)db$$

$$(*) = \mathbb{P}(X_t - X_1 < -b) = \int_{-\infty}^{-b} \phi_{t-1}(x)dx = \int_b^\infty \phi_{t-1}(x)dx$$

Converting to normal with $y = \frac{x}{\sqrt{t-1}}$, this is

$$(*) = \int_{\frac{b}{\sqrt{t-1}}}^\infty \phi_1(y)dy$$

So, the answer is

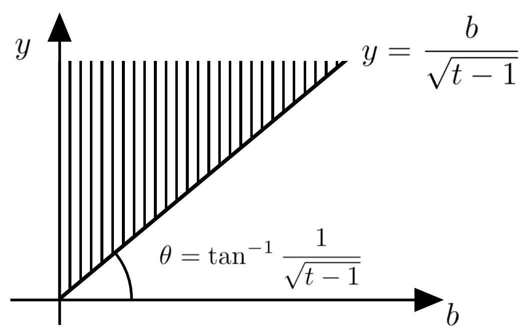
$$ans = 4\int_{b=0}^{\infty}\int_{y=\frac{b}{\sqrt{t-1}}}^{\infty} \phi_1(b)\phi_1(y)\,dydb$$

$$\phi_1(b)\phi_1(y) = \frac{1}{\sqrt{2\pi}}e^{-b^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{2\pi}e^{-\frac{b^2+y^2}{2}}$$

Convert to polar coordinates: $b^2 + y^2 = r^2, dbdy = rdrd\theta$. Then

$$ans = 4\int_{\tan^{-1}\frac{1}{\sqrt{t-1}}}^{\pi/2}\int_0^{\infty}\frac{1}{2\pi}e^{-r^2/2}rdrd\theta$$

The limits of integration are given by the picture



The integral is easy to calculate

$$\int_0^{\infty}\frac{4}{2\pi}e^{-r^2/2}r\,dr = \frac{2}{\pi}$$

So,

$$ans = \int_{\tan^{-1}\frac{1}{\sqrt{t-1}}}^{\pi/2}\frac{2}{\pi}\,d\theta = \frac{2}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{t-1}}\right] = 1 - \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{t-1}}$$

The expression is 1- (something) where the "something" is the probability that the event does not occur. So,

$$\frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{t-1}} = \mathbb{P}(X_s \neq 0 \text{ for all } 1 < s < t)$$