8.3. **fractal dimension of zero set.** Today we calculated the fractal dimension of the zero set.

First, I went over the calculation from last time. Suppose that X_t is Brownian motion which is centered with drift zero (CDZ). Then

$$\mathbb{P}(X_s = 0 \text{ for some } 1 < s \le t \,|\, X_0 = 0) = \boxed{1 - \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{t-1}}}$$

This can be rephrased in terms of the *zero set* Z of X_t which is defined to be the set of all times t at which $X_t = 0$:

$$Z := \{t \,|\, X_t = 0\}$$

With this definition, the statement $X_s = 0$ is the same as $s \in Z$. So, the calculation above can be written as

$$\mathbb{P}(Z \cap (1,t] \ne \emptyset) = 1 - \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{t-1}}$$

For example, if $t = 2$ we get $\tan^{-1}(1) = \pi/4$. So,

$$\mathbb{P}(Z \cap (1,2] \ne \emptyset) = \frac{1}{2}$$

$$\mathbb{P}(Z \cap (1,2] = \emptyset) = \frac{1}{2}$$

To calculate other probabilities we need to rescale Z.

The first thing I pointed out about rescaling is that the set Z is independent of the variance σ^2. Remember that we are assuming that X_t is centered with drift zero. The variance σ^2 is arbitrary. But then

$$X_t = \sigma W_t$$

where W_t is standard Brownian motion. But, these have the same zero set:

$$X_t = 0 \Leftrightarrow W_t = 0$$

$$(\text{zero set of } X_t) = (\text{zero set of } W_t)$$

So, Z is the zero set of standard Brownian motion. The value of σ^2 is irrelevant. It only tells how far away from zero you go, not when you hit zero.

8.3.1. *Z is a fractal.* This means that Z is "self-similar." If you scale it up, it looks the same.

Theorem 8.8. *If $a > 0$ then X_{at} is Brownian motion (CDZ) with variance $a\sigma^2$.*

Proof. $Y_t := X_{at}$ is continuous and memoryless (in the sense that the increments of Y_t on disjoint intervals are independent) and $Y_0 = X_0 = 0$. So, we just have to show that the increments are normal:

$$Y_t - Y_s = X_{at} - X_{as} \sim N(0, \sigma^2(at - as)) = N(0, a\sigma^2(t - s))$$

So, Y_t is Brownian motion with no drift and variance $a\sigma^2$. □

Corollary 8.9. *$\frac{1}{a}Z$ has the same probability distribution as Z if $a > 0$.*

Remark 8.10. First, I explained that Z is a random set. Every time you do the experiment you will observe a different zero set. The probability distribution tell the probability that Z will look a certain way. But if you scale it up or down, it will give the same sets with the same probabilities. That is what this corollary says.

Proof.

$$Y_t = 0 \quad \Leftrightarrow \quad X_{at} = 0 \Leftrightarrow at \in Z \Leftrightarrow t \in \tfrac{1}{a}Z.$$

But Y_t is Brownian, CDZ and variance $a\sigma^2$. This means that $Y_t = a\sigma^2 W_t$. As I explained earlier, this means that the zero set of Y_t is the same as the zero set of W_t. In other words, it is another Z (a variable with the same distribution as Z). So, Z and $\frac{1}{a}Z$ have the same distribution. □

8.3.2. *probability of gaps in Z.* We already know that

$$\mathbb{P}(Z \cap (1, 2] \neq \emptyset) = \frac{1}{2}$$

But now we know that, for any positive a, Z has the same distribution as $\frac{1}{a}Z$. So,

$$\mathbb{P}(\tfrac{1}{a}Z \cap (1, 2] \neq \emptyset) = \frac{1}{2}$$

If we multiply by a we get:

$$\mathbb{P}(Z \cap (a, 2a] \neq \emptyset) = \frac{1}{2}.$$

For example, this implies that

$$\mathbb{P}(Z \cap (1, 4] = \emptyset) = \mathbb{P}(Z \cap (1, 2] = \emptyset)\mathbb{P}(Z \cap (2, 4] = \emptyset) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

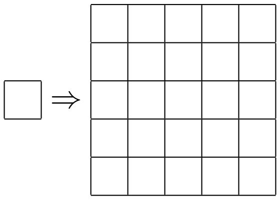
In fact, we can now calculate the probability that Z will meet any interval. If $0 < a < b$ then, scalling by $1/a$, we get

$$\mathbb{P}(Z \cap (a, b] \neq \emptyset) = \mathbb{P}(Z \cap (1, b/a] \neq \emptyset) = 1 - \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{\frac{b}{a} - 1}}$$

With this formula we can calculate the fractal dimension of Z. First I explained the definition:

8.3.3. *fractal dimension.* I explained it twice. First by scaling the set up and then by scaling the units down.

a) by scaling up. If you take a square and scale it up by a factor of 5 you get $25 = 5^2$ squares. So, the dimension of the square is 2 (the exponent of the scaling factor).



If you take a cube and scale it up by a factor of 5 in every direction you get $125 = 5^3$ squares. So, the dimension of the cube is 3.

The *Cantor set* is the set of real number from 0 to 1 minus the middle third open interval, and then you keep deleting the middle third interval:

$$C = [0,1] - \left(\frac{1}{3}, \frac{2}{3}\right) - \left(\frac{1}{9}, \frac{2}{9}\right) - \left(\frac{7}{9}, \frac{8}{9}\right) - \left(\frac{1}{27}, \frac{2}{27}\right) - \cdots$$

Since an infinite union of open sets is open, C is closed and thus compact. (A closed bounded subset of $\mathbb{R}^n$ is compact.)

If you scale C up by a factor of 3 then you get 2 Cantor sets since the intervals $[1/3, 2/3]$ and $[2/3, 1]$ look like the whole thing. The dimension of C is the exponent that you need to raise the scaling factor 3 by to get 2:

$$2 = 3^D \Rightarrow \ln 2 = D \ln 3$$

$$D = \frac{\ln 2}{\ln 3} \doteq .631$$

b) by scaling down the units

Definition 8.11. The *box dimension* of a bounded set $A \subseteq \mathbb{R}^d$ is the infimum of all $D \geq 0$ so that, as $n \to \infty$, the number of little cubes with sides $1/n$ needed to cover A is

$$< C\, n^D$$

for some constant C.

For example, a 1×1 square is covered with 5^2 little $\frac{1}{5} \times \frac{1}{5}$ squares.

8.3.4. *dimension of Z.* I explain why the dimension of Z is exactly $D = 1/2$. First I had to make it bounded by taking

$$Z_1 = Z \cap (0,1] \subseteq (0,1].$$

Now, take n large. We cut up the interval $(0,1]$ into n subintervals $(0,1/n],(1/n,2/n],\cdots$:

$$\left(\frac{k}{n},\frac{k+1}{n}\right],\quad k=0,1,2,\cdots,n-1.$$

The question is: How many of these intervals do I need to cover Z?

The number N of little 1-cubes (intervals) that I need is the sum of indicator functions:

$$N=\sum_{k=0}^{n-1} I_k$$

where

$$I_k=\begin{cases}1 & \text{if } Z\cap\left(\frac{k}{n},\frac{k+1}{n}\right]\neq\emptyset\\ 0 & \text{if not}\end{cases}$$

The indicator function I_k gives a 1 every time I need the interval $\left(\frac{k}{n},\frac{k+1}{n}\right]$ to cover Z. Therefore, the sum of the I_k is the number of intervals that I need.

Since this is random, we calculate the expected value of N:

$$\mathbb{E}(N)=\mathbb{E}\left(\sum I_k\right)=\sum_{k=0}^{n-1}\mathbb{E}(I_k)$$

since $\mathbb{E}$ is linear. But, as I told you earlier in the semester, the expected value of the indicator function is the probability:

$$\mathbb{E}(I_k)=\mathbb{P}\left(Z\cap\left(\frac{k}{n},\frac{k+1}{n}\right]\neq\emptyset\right)$$

which we can compute by scaling by a factor of $\frac{n}{k}$:

$$=\mathbb{P}\left(Z\cap\left(1,1+\frac{1}{k}\right]\neq\emptyset\right)=1-\frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{\frac{1}{k}-1}}$$

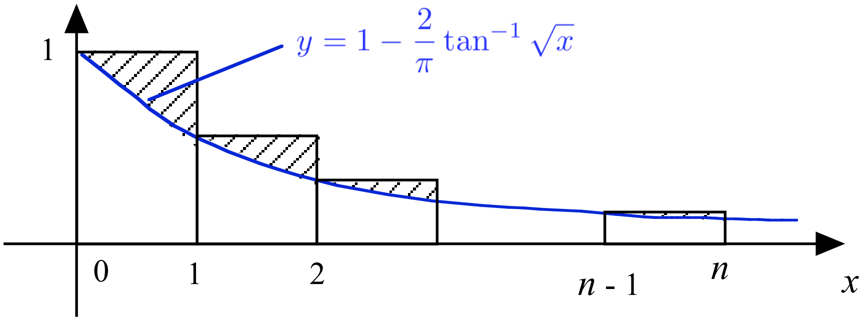
$$=1-\frac{2}{\pi}\tan^{-1}\sqrt{k}$$

So, the expected number of intervals that we need is

$$\mathbb{E}(N)=\sum_{k=0}^{n-1}1-\frac{2}{\pi}\tan^{-1}\sqrt{k}$$

This is equal to the integral (shown in blue in the figure) plus an error ϵ (shaded region) which is $\epsilon\approx\frac{1}{2}$ and definitely $\epsilon<1$ since the little triangles can be stacked up inside the first rectangle.

$$\mathbb{E}(N)=\int_0^n 1-\frac{2}{\pi}\tan^{-1}\sqrt{x}\,dx+\epsilon$$



Doing the integral gives:

$$\mathbb{E}(N) = \epsilon + n - \frac{2}{\pi}(n+1)\tan^{-1}\sqrt{n} + 2\sqrt{n}.$$

Now we use the approximation:

$$\tan^{-1}\sqrt{n} \approx \frac{\pi}{2} - \frac{1}{\sqrt{n}}$$

for n large. This comes from the Taylor series:

$$\tan^{-1}\frac{1}{x} = \frac{\pi}{2} - \tan^{-1}x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

So,

$$\tan^{-1}\sqrt{n} = \frac{\pi}{2} - \frac{1}{\sqrt{n}} + \frac{1}{3n^{3/2}} - \frac{1}{5n^{5/2}} + \cdots$$

Using this approximation you get

$$\mathbb{E}(N) \approx \epsilon + n - \frac{2}{\pi}(n+1)\left(\frac{\pi}{2} - \frac{1}{\sqrt{n}}\right) + 2\sqrt{n}$$

$$\epsilon + n - (n+1) + \frac{2}{\pi}\frac{n+1}{\sqrt{n}} + 2\sqrt{n}$$

$$= \underbrace{\epsilon - 1}_{\text{bounded}} + \underbrace{\frac{2}{\pi\sqrt{n}}}_{\to 0} + \underbrace{\left(\frac{2}{\pi} + 2\right)}_{C}\underbrace{\sqrt{n}}_{n^D}$$

This is Cn^D where $D = \frac{1}{2}$. Therefore,

$$\boxed{\dim Z = \frac{1}{2}}$$