8.4. **Brownian motion in $\mathbb{R}^d$ and the heat equation.** The heat equation is a partial differential equation. We are going to convert it into a probabilistic equation by reversing time. Then we can using stopping time.

8.4.1. *definition.*

Definition 8.12. *d-dimensional Brownian motion* with drift $\mu = \mathbf{0} \in \mathbb{R}^d$ and variance σ^2 is a vector valued stochastic process

$$\mathbf{X}_t \in \mathbb{R}^d, \quad t \in [0, \infty)$$
$$\mathbf{X}_t = (X_t^1, X_t^2, \cdots, X_t^d)$$

so that

(1) $\mathbf{X}_t$ is continuous
(2) The increments $\mathbf{X}_{t_i} - \mathbf{X}_{s_i}$ of $\mathbf{X}_t$ on disjoint time intervals $(s_i, t_i]$ are independent.
(3) Each coordinate of the increment is normal with the same variance:
$$X_t^i - X_s^i \sim N(0, \sigma^2(t-s))$$
and they are independent.

This implies that the density function of $\mathbf{X}_t - \mathbf{X}_s$ is a product of normal density functions:

$$f_{t-s}(\mathbf{x}) = f_{t-s}(x_1) \cdots f_{t-s}(x_d) = \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-x_i^2/2\sigma^2(t-s)}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}^d} e^{-||\mathbf{x}||^2/2\sigma^2(t-s)}$$

since $\prod e^{whatever} = e^{\sum whatever}$ and

$$\sum_{i=1}^{d} x_i^2 = ||\mathbf{x}||^2.$$

Since this is the density of the increment $\mathbf{X}_t - \mathbf{X}_s$, it gives the transition "matrix"

$$p_{\Delta t}(\mathbf{x}, \mathbf{y}) = f_{\Delta t}(\mathbf{y} - \mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}^d} e^{-||\mathbf{y}-\mathbf{x}||^2/2\sigma^2\Delta t}$$

The point is that $||\mathbf{y} - \mathbf{x}|| = ||\mathbf{x} - \mathbf{y}||$. So,

$$p_{\Delta t}(\mathbf{x}, \mathbf{y}) = p_{\Delta t}(\mathbf{y}, \mathbf{x}).$$

In other words, Brownian motion (with zero drift) is a symmetric process. When you reverse time, it is the same. (It is also obvious that if there is a drift μ, the time reversed process will have drift $-\mu$.)

8.4.2. *diffusion (the heat equation).* If we have a large number of particles moving independently according to Brownian motion then the density of particles at time t becomes a deterministic process called *diffusion*. It satisfies a differential equation called the *heat equation*. When we reverse time, we will get a probabilistic version of this equation called the "backward equation."

Let $f(\mathbf{x})$ be the density of particles (or heat) at position $\mathbf{x}$ at time t. Then we have the Chapman-Kolmogorov equation, also called the *forward equation*:

$$f_{t+\Delta t}(\mathbf{y}) = \int_{\mathbb{R}^d} f_t(\mathbf{x}) p_{\Delta t}(\mathbf{x}, \mathbf{y}) \underbrace{d\mathbf{x}}_{dx_1 \cdots dx_d}$$

But, $p_{\Delta t}(\mathbf{x}, \mathbf{y}) = p_{\Delta t}(\mathbf{y}, \mathbf{x})$ since Brownian motion is symmetric when $\mu = 0$. So,

$$f_{t+\Delta t}(\mathbf{y}) = \int_{\mathbb{R}^d} f_t(\mathbf{x}) p_{\Delta t}(\mathbf{y}, \mathbf{x})\, d\mathbf{x}$$

Since equations remain true when you change the names of the variables, this equation will still hold if I switch $\mathbf{x} \leftrightarrow \mathbf{y}$. This gives the *backward equation*:

$$f_{t+\Delta t}(\mathbf{x}) = \underbrace{\int_{\mathbb{R}^d} f_t(\mathbf{y}) p_{\Delta t}(\mathbf{x}, \mathbf{y})\, d\mathbf{y}}_{\text{This is an expected value.}}$$

The RHS is an expected value since it is the sum of $f(\mathbf{y})$ times it probability. Since $\mathbf{x}$ moves to $\mathbf{y}$ in time Δt, $\mathbf{y} = \mathbf{X}_{t+\Delta t}$.

$$\int_{\mathbb{R}^d} f_t(\mathbf{y}) p_{\Delta t}(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = \mathbb{E}_t^{\mathbf{x}}(f_t(\mathbf{X}_{t+\Delta t})) = \mathbb{E}(f_t(\mathbf{X}_{t+\Delta t}) \,|\, \mathbf{X}_t = \mathbf{x})$$

Where $\mathbb{E}_t^{\mathbf{x}}$ means expectation is conditional on $\mathbf{X}_t = \mathbf{x}$. In words:

$$\textit{Future density at the present location } \mathbf{x}$$

$$= \textit{expected value of the present density at the future location } \mathbf{y}$$

using the following interpretation of "present" and "future"

	time	*location*
present	t	$\mathbf{x}$
future	$t + \Delta t$	$\mathbf{y}$

8.4.3. *Calculate* $\frac{\partial}{\partial t} f_t$. I want to calculate

$$\frac{\partial}{\partial t} f_t(\mathbf{x}) = \lim_{\Delta t \to 0} \frac{f_{t+\Delta t}(\mathbf{x}) - f_t(\mathbf{x})}{\Delta t}.$$

Using the backward equation, this is

$$= \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^{\mathbf{x}}(f_t(\mathbf{X}_{t+\Delta t})) - f_t(\mathbf{x})}{\Delta t}$$

To figure this out I used the Taylor series. Here it is when $d = 1$.

$$f_t(X_{t+\Delta t}) = f_t(X_t) + f_t'(X_t)\Delta X + \frac{1}{2} f_t''(X_t)((\Delta X)^2) + O((\Delta X)^3)$$

Here $f_t' = \frac{\partial}{\partial x} f_t$. The increment in X is

$$\Delta X = X_{t+\Delta t} - X_t \sim N(0, \sigma^2 \Delta t)$$

This means that

$$\mathbb{E}((\Delta X)^2) = \sigma^2 \Delta t.$$

In other words, $(\Delta X)^2$ is expected to be on the order of Δt. So, $(\Delta X)^3$ is on the order of $(\Delta t)^{3/2}$. So,

$$\frac{\mathbb{E}(\epsilon)}{\Delta t} = \frac{\mathbb{E}(O((\Delta X)^3)}{\Delta t} \to 0 \qquad \text{as } \Delta t \to 0$$

Taking expected value and substituting $X_t = x$ we get:

$$\mathbb{E}_t^x(f_t(X_{t+\Delta t})) - f_t(x) = f_t'(x) \underbrace{\mathbb{E}(\Delta X)}_{=-\mu=0} + \frac{1}{2} f_t''(x) \mathbb{E}_t^x((\Delta X)^2) + \mathbb{E}(\epsilon)$$

$$= \frac{1}{2} f_t''(x)\sigma^2 \Delta t + \mathbb{E}(\epsilon)$$

$$\frac{\mathbb{E}_t^x(f_t(X_{t+\Delta t})) - f_t(x)}{\Delta t} = \frac{1}{2} f_t''(x)\sigma^2 + \underbrace{\frac{\mathbb{E}(\epsilon)}{\Delta t}}_{\to 0}$$

So,

$$\boxed{\frac{\partial}{\partial t} f_t(x) = \frac{\sigma^2}{2} f_t''(x)}$$

In higher dimensions we get the following

$$\boxed{\frac{\partial}{\partial t} f_t(\mathbf{x}) = \frac{\sigma^2}{2} \Delta f_t(\mathbf{x})}$$

where Δ is the *Laplacian*:

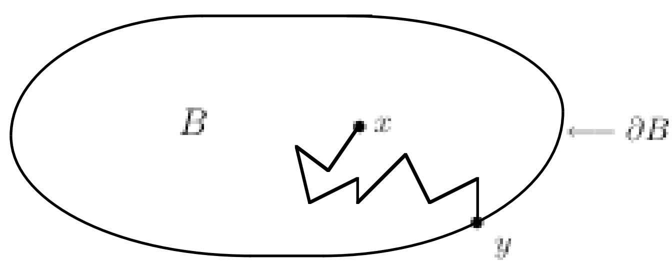
$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

This follows from the multivariable Taylor series:

$$f_t(\mathbf{X}_{t+\Delta t}) = f_t(\mathbf{X}_t) + \sum_i \frac{\partial f_t(\mathbf{X}_t)}{\partial x_i}\Delta X_t^i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f_t(\mathbf{X}_t)}{\partial x_i \partial x_j}\Delta X_t^i \Delta X_t^j + \epsilon$$

Since $\mathbb{E}(\Delta X_t^i) = -\mu_i = 0$, the Σ_i terms have expected value zero and the $\Sigma_{i,j}$ terms also have zero expected value when $i \neq j$. This leaves the $\Sigma_{i,i}$ terms which give the Laplacian. I pointed out in class that $\mathbb{E}(\Delta X_t^i) = -\mu_i$ because we are using the backward equation.

8.4.4. *boundary values.* Now we want to solve the boundary valued problem, or at least convert it into a probability equation. Suppose we have a bounded region B and we heat up the boundary ∂B.



Let

$$f(x) = \text{ current temperature at } x \in B$$
$$g(y) = \text{ current temperature at } y \in \partial B$$

Suppose the $g(y)$ is fixed for all $y \in \partial B$. This is the heating element on the outside of your oven. The point x is in the inside of your oven. The temperature $f(x)$ is changing according to the heat equation:

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\Delta f.$$

We want to calculate $u(t,x) = f_t(x)$, the temperature at time t. We also want the *equilibrium* temperature $v(x) = f_\infty(x)$. When the oven has been on for a while it stabilizes and

$$\frac{\partial}{\partial t} f_\infty(x) = 0$$

which forces

$$\boxed{\Delta f_\infty(x) = 0}$$

If we use the backward equation we can use the stopping time $T=$ the first time you hit ∂B. Taking it to be a stopping time means that the boundary is "sticky" like flypaper. The particle x bounces around inside the region B until it hits the boundary ∂B and then it stops. (You can choose your stopping time to be anything that you want.) Then the backward equation, using OST, is:

$$v(t,x) = \mathbb{E}^x(g(X_T)I(T \le t) + f(X_T)I(t < T))$$

Here $I(T \le t)$ is the indicator function for the event that $T \le t$. Multiplication by this indicator function is the same as the condition "if $T \le t$." The equilibrium temperature is given by

$$f_\infty(x) = v(x) = \mathbb{E}^x(g(X_T))$$

This is an equation we studied before. $v(x)$ is the *value function*. It gives your expected payoff if you start at x and use the optimal strategy. $g(x)$ is the payoff function. X_T is the place that you will eventually stop if you use your optimal strategy which is the formula for the stopping time T.

I gave one really simple example to illustrate this concept.

Example 8.13. You give a professional gambler \$$x$ and send him to a casino to play until he loses (when he has \$0) or wins (by geting $y = \$10^3$). The gambler gets a fee of \$$a$ if he loses and \$$b$ if he wins. The question is: What is his expected payoff?

T = stopping time is the first time that $X_T = 0$ or y. We have $B = [0, y]$ with boundary $\partial B = \{0, y\}$ and boundary values:

$$g(0) = a, \quad g(y) = b.$$

The expected payoff, starting at x, is

$$v(x) = \mathbb{E}^x(g(X_T)) = a\mathbb{P}^x(X_T = 0) + b\mathbb{P}^x(X_T = y).$$

But, $\mathbb{E}^x(X_T) = X_0 = x$ by the Optimal Sampling Theorem. This is:

$$y\mathbb{P}^x(X_T = y) = x$$

making $\mathbb{P}^x(X_T = y) = x/y$ and

$$\mathbb{P}^x(X_T = 0) = 1 - \frac{x}{y}.$$

Therefore,

$$v(x) = a\left(1 - \frac{x}{y}\right) + b\frac{x}{y}$$

This is a linear function which is equal to a at $x = 0$ and $v(y) = b$. While this was fun, this is an example where the boring analytic method is actually much more efficient: The heat equation says

$$\Delta v(x) = v''(x) = 0$$

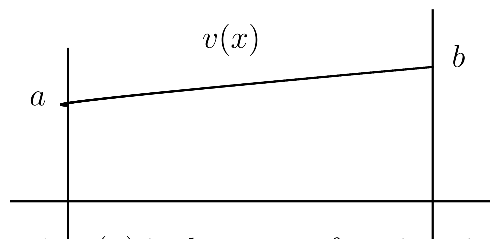


FIGURE 1. $v(x)$ is the convex function given by the convex hull of the points $(0, a), (y, b)$ which are the given payoffs.

In other words, the derivative $v'(x)$ is constant and $v(x)$ is a straight line function. With the given values $v(0) = a, v(y) = b$ we get the answer right away.