8.5. **Recurrence and transience.** The question is: Does Brownian motion make particle go off to ∞?

(1) Set up the probabilistic equation
(2) Convert to a differential equation by time reversal
(3) Solve the differential equation
(4) Reinterpret probabilitically

8.5.1. *set up.* We start at a point x which is off center between two circles (or spheres in dimensions ≥ 3)

$$\mathbf{x} \in B = \{\mathbf{x} \in \mathbb{R}^d \,|\, R_1 < ||x|| < R_2\}$$

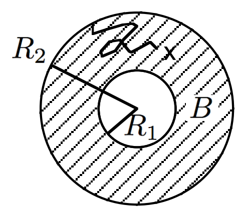

FIGURE 2. What is the probability that $\mathbf{x}$ will reach the outer circle before reaching the inner circle?

Take the stopping time T to be the first time that $\mathbf{X}_T \in \partial B$ given that $\mathbf{X}_0 = \mathbf{x} \in B$. We now want to know: What is the probability that $||\mathbf{X}_T|| = R_2$?

$$f(\mathbf{x}) = \mathbb{P}^{\mathbf{x}}(||\mathbf{X}_T|| = R_2) = ?$$

But this is the same as

$$f(\mathbf{x}) = \mathbb{E}^{\mathbf{x}}(g(\mathbf{X}_T))$$

where $g(\mathbf{y})$ is given by

$$g(\mathbf{y}) = \begin{cases} 1 & \text{if } ||\mathbf{y}|| = R_2 \\ 0 & \text{if } ||\mathbf{y}|| = R_1 \end{cases}$$

The notation $\mathbb{P}^{\mathbf{x}}$ and $\mathbb{E}^{\mathbf{x}}$ means probability and expectation conditional on $\mathbf{X}_0 = \mathbf{x}$.

8.5.2. *differential equation.* By the time reversal argument explained last time, $f(\mathbf{x})$ is the solution of the differential equation

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} = 0$$

on B with boundary condition

$$f(\mathbf{y}) = \begin{cases} 1 & \text{if } ||\mathbf{y}|| = R_2 \\ 0 & \text{if } ||\mathbf{y}|| = R_1 \end{cases}$$

Since everything is rotationally symmetric, we know that the solution will be a function of $||\mathbf{x}||$. It is also a function of $z = ||\mathbf{x}||^2 = \sum x_i^2$ which I much prefer since it has no nasty square roots.

$$f(\mathbf{x}) = \phi(z) = \phi\left(\sum x_i^2\right)$$

$$\frac{\partial f}{\partial x_i} = 2x_i\phi'(z)$$

$$\frac{\partial^2 f}{\partial x_i^2} = 2\phi'(z) + 4x_i^2\phi''(z)$$

Sum over all $i = 1, 2, \cdots, d$ to get

$$\boxed{\Delta f(\mathbf{x}) = 2d\phi'(z) + 4z\phi''(z) = 0}$$

8.5.3. *solution of diffeq.* Put $\psi(z) = \phi'(z)$. Then the equation is

$$2d_{im}\psi(z) + 4z\psi'(z) = 0$$

where I wrote the dimension d as "d_{im}" temporarily so that I can write this as:

$$4z\frac{d\psi}{dz} = -2d_{im}\psi$$

Separate the variables:

$$\frac{d\psi}{\psi} = -\frac{d_{im}}{2}\frac{dz}{z}$$

Integrate both sides to get

$$\ln\psi = -\frac{d_{im}}{2}\ln z + C_0$$

or (dropping the $_{im}$ subscript)

$$\psi = \phi' = K_0 z^{-d/2}$$

where $K_0 = e^{C_0}$. Integrate to get $f = \phi$. When $d \neq 2$ this is:

$$f(\mathbf{x}) = \phi(z) = K_0\frac{2z^{(2-d)/2}}{2-d} + C$$

Letting $K := 2K_0/(2-d)$ we can write this as:

$$\boxed{f(\mathbf{x}) = K||\mathbf{x}||^{2-d} + C}$$

Now we put in the boundary conditions. First, $f(\mathbf{x}) = 0$ if $||\mathbf{x}|| = R_1$. This gives

$$0 = KR_1^{2-d} + C$$

or

$$C = -KR_1^{2-d}$$

The other boundary condition is $f(\mathbf{x}) = 1$ when $||\mathbf{x}|| = R_2$. This gives

$$1 = KR_2^{2-d} - KR_1^{2-d}$$

or

$$K = \frac{1}{R_2^{2-d} - R_1^{2-d}}$$

So, the solution (for $d \neq 2$) is

$$\boxed{f(\mathbf{x}) = \frac{||\mathbf{x}||^{2-d} - R_1^{2-d}}{R_2^{2-d} - R_1^{2-d}}}$$

If we put $d = 2$ we get $\frac{0}{0}$ and we can get the answer in the book by taking the limit as $d \to 2$ using l'Hostipal's rule. (That's called "dimensional regularization." It isn't rigorous but it works.)

Since each term has the form C^{2-d}, when you differentiate with respect to d, you get

$$\left(C^{2-d}\right)' = (-\ln C)\underbrace{C^{2-d}}_{\to 1}$$

which goes to $-\ln C$ as $d \to 2$. So, the answer, for $d = 2$, is:

$$\boxed{f(\mathbf{x}) = \frac{-\ln||\mathbf{x}|| + \ln R_1}{-\ln R_2 + \ln R_1}}$$

8.5.4. *interpret solution.* Remember that $f(\mathbf{x})$ is the probability that $||\mathbf{x}||$ will reach R_2 before it reaches R_1. So, we want to take the limit as $R_1 \to 0$ and $R_2 \to \infty$.

a) ($R_2 \to \infty$) Suppose that $d = 3$. Then the first boxed equation is

$$f(\mathbf{x}) = \frac{\frac{1}{||\mathbf{x}||} - \frac{1}{R_1}}{\frac{1}{R_2} - \frac{1}{R_1}}$$

When R_2 goes to ∞, $1/R_2 \to 0$. So, $f(\mathbf{x})$ becomes:

$$\lim_{R_2\to\infty} f(\mathbf{x}) = \frac{\frac{1}{||\mathbf{x}||} - \frac{1}{R_1}}{-\frac{1}{R_1}} = 1 - \frac{R_1}{||\mathbf{x}||}$$

Since $R_1 < ||\mathbf{x}|| < R_2$, this is between 0 and 1. This means the particle has a chance to go to infinity. So, eventually it will with probability one. (If it hits the small circle then the particle does not actually stop. It continues and has the same chance to go to ∞ before it hits a smaller circle around 0.) So, Brownian motion in dimension 3 is *transient*. The same thing happens for any $d > 2$:

$$\lim_{R_2\to\infty} f(\mathbf{x}) = 1 - \frac{R_1}{||\mathbf{x}||^{d-2}}$$

So, Brownian motion is *transient* for all $d > 2$.

Now suppose that $d = 2$. When we take the limit as $R_2 \to \infty$,

$$\ln R_2 \to \infty$$

So,

$$\lim_{R_2\to\infty} f(\mathbf{x}) = 0$$

The particle will never go to infinity. It will keep returning to the circle of radius R_1 about the origin no matter how small $R_1 > 0$ is. And the same is true for any point in the plane. Brownian motion will hit any circle around any point infinitely many times with probability one. So, Brownian motion in $\mathbb{R}^2$ is called *(neighborhood) recurrent*.

b) ($R_1 \to 0$) When $d > 2$,

$$R_1^{2-d} = \frac{1}{R_1^{d-2}} \to \infty$$

So, $f(\mathbf{x}) \to 1$. Similarly, for $d = 2$,

$$\ln R_1 \to -\infty$$

So, $f(\mathbf{x}) \to 1$.

This means that, for $d \geq 2$, the probability is zero that the particle will ever return to the origin. So, when $d = 2$ Brownian motion is not actually recurrent. It is only neighborhood recurrent.

When $d = 1$,

$$\lim_{R_1\to 0} f(\mathbf{x}) = ||\mathbf{x}||/R_2 < 1$$

The particle has a chance to go to the origin and therefore it eventually will with probability one. Then it will keep coming back because it can't avoid probability one events. And this is true for every point on the line. Brownian motion on a line goes to every point on the line infinitely often with probability one. So, Brownian motion on a line is recurrent.