

MATH 56A SPRING 2008  
STOCHASTIC PROCESSES

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## 9. STOCHASTIC INTEGRATION

This is integration with respect to Brownian motion. The main goal is to prove and understand the Black-Scholes formula for pricing stock options.

I gave a “preview” of this last week. I emphasized the key mathematical concept which is called “quadratic variation.” Another thing I mentioned was the change of notation. Since  $\Delta = \sum \frac{\partial}{\partial x_i}$  is the Laplacian, I will use  $\delta$  to indicate increments. E.g.,  $\delta t = t_{i+1} - t_i$ .

**9.0. Concept, Itô’s formula, quadratic variation.** First, I explained the idea behind the stochastic integral emphasizing the predictability of  $Y_t$  and the vanishing cross term argument.

9.0.1. *idea of stochastic integral.* We want to define and understand the stochastic integral:

$$Z_t = \int_0^t Y_s dX_s$$

The interpretation is:

$X_t$  = price of one share of stock at time  $t$ .

$Y_t$  = number of shares of this stock that you hold at time  $t$ .

We will assume that  $Y_t \in \mathbb{R}$  is a real number. If you use computer records instead of paper certificates, you can hold fractional shares of stock. Also, this number can be negative which would mean you owe that many shares to someone.

I also assume that  $Y_t$  is “predictable” (also called “previsible”) which means  $\mathcal{F}_t$ -measurable and left continuous. Some people write “ $\mathcal{F}_{t-}$ -measurable.”

**Definition 9.1.** *Predictable* means that  $Y_t$  depends on all of the information you have *up to but not including time  $t$* . I.e.,  $Y_t$  is  $\mathcal{F}_s$ -measurable for all  $s < t$ .

The assumption of predictability of the integrand  $Y_t$  is somewhat controversial. Critics ask, e.g., if  $Y_0$  is measurable. (To fix this, I will just add the assumption that  $Y_0$  is  $\mathcal{F}_0$ -measurable.) Predictability of  $Y_t$  is an essential part of my philosophical understanding of this subject. More on this later.

Since  $Y_t$  varies with time, you are constantly trading stocks. Assume that there is no commission. The question is:

Q: How much money did you make buying and selling stock?

To answer this, look at the time interval  $(t, t + \delta t]$ . The price of your stock increased by  $\delta X_t = X_{t+\delta t} - X_t$ . You held  $Y_t$  shares of stock

throughout this time interval (and you might have bought/sold some during the interval but we are going to ignore this). So, the amount of money you made (your change in equity) is approximately  $Y_t \delta X_t$ . Your total net gain is approximately the sum  $\sum Y_t \delta X_t$ . To get the precise answer, you take the limit as  $\delta t$  goes to 0.

**Definition 9.2.** The *stochastic integral* of  $Y_t$  is

$$Z_t = \int_0^t Y_s dX_s := \lim_{\delta t \rightarrow 0} \sum Y_s \delta X_s$$

This is Kiyoshi Itô's definition of a stochastic integral. There is another definition due to Stratonovich which uses

$$\frac{Y_t + Y_{t+\delta t}}{2}$$

instead of  $Y_t$ . (You can also take  $Y_{t+\frac{1}{2}\delta t}$ ). The Stratonovich integral is not natural since it uses information from the future. However, it has better mathematical properties. For example, it is defined on any smooth manifold whereas the Itô integral is not.

9.0.2. *Itô's formula.* I explained the formula without proof. We will prove it later.

Here is the example I did in class:

$$Z_t = \int_0^t X_s dX_s = ?$$

This formula represents a strategy that you could actually implement. It says you should hold the number of shares equal to the price of each share. So, if it goes up, you buy more; if it goes down you sell. The usual method of integration says that you integrate the function  $x$  and get

$$f(x) = \frac{x^2}{2}.$$

So,

$$\int_0^t x dx = f(t) - f(0) = \frac{t^2}{2}$$

Assuming that  $X_t$  is Brownian motion with  $\mu = 0$  and variance  $\sigma^2$ , the value of the stochastic integral is

$$Z_t = \int_0^t X_s dX_s = \frac{X_t^2 - X_0^2}{2} + \boxed{-\frac{\sigma^2 t}{2}}$$

Itô's formula for the error term is

$$error = -\frac{\sigma^2}{2} \int_0^t f''(s) ds$$

For our example,  $f(x) = x^2/2$ . So,  $f''(s) = 1$  making the error terms  $-\sigma^2 t/2$ .

The interpretation of this formula is: You can't make money with this or any other predictable process. This is because

$$\mathbb{E}(X_t^2 - X_0^2) = \sigma^2 t$$

So, we are subtracting the expected value making  $Z_t$  into a martingale.

(For the Stratonovich definition of the stochastic integral, you would cut the error term in half. Since the error is negative, the payoff increases. This means you can make a profit if you can actually implement that definition and find the future value of the stock half a second before you are supposed to know it.)

9.0.3. *quadratic variation.* Since  $Z_t$  is a martingale and  $Z_0 = 0$ ,  $\mathbb{E}(Z_t) = 0$ . The next question is:

What is  $\mathbb{E}(Z_t^2)$ ?

We assume that  $X_t$  is Brownian motion with zero drift.

To answer this, take the square of the approximation to  $Z_t$ :

$$\begin{aligned} & \left( \sum_i Y_{t_i} \delta X_{t_i} \right)^2 \\ &= \sum_i Y_{t_i}^2 \underbrace{(\delta X_{t_i})^2}_{\mathbb{E}=\sigma^2 \delta t} + 2 \sum_{i < j} \underbrace{Y_{t_i} Y_{t_j} \delta X_{t_i} \delta X_{t_j}}_{\mathcal{F}_{t_j}\text{-measurable}} \underbrace{\delta X_{t_j}}_{\mathbb{E}=0} \end{aligned}$$

The first term has expected value

$$\sum_i \mathbb{E}(Y_{t_i}^2 (\delta X_{t_i})^2) = \sum_i \mathbb{E}(Y_{t_i}^2) \mathbb{E}(\delta X_{t_i})^2 = \sum_i \mathbb{E}(Y_{t_i}^2) \sigma^2 \delta t$$

where we use the fact that  $Y_{t_i}$  and  $\delta X_{t_i} = X_{t_{i+1}} - X_{t_i}$  are independent. If we take the limit as  $\delta t$  goes to zero we get

$$\lim_{\delta t \rightarrow 0} \sum_i \mathbb{E}(Y_{t_i}^2 (\delta X_{t_i})^2) = \int_0^t \mathbb{E}(Y_s) \sigma^2 ds.$$

The cross term  $Y_{t_i} Y_{t_j} \delta X_{t_i} \delta X_{t_j}$  is a product of 4 random variables. At time  $t_j$  we will know the value of the first three terms since  $t_i < t_j$ . The 4th term will have expected value 0 since  $\mu = 0$ :

$$\mathbb{E}(\delta X_{t_j} | \mathcal{F}_{t_j}) = \mathbb{E}(X_{t_{j+1}} - X_{t_j} | \mathcal{F}_{t_j}) = \mu \delta t = 0.$$

Therefore,

$$\mathbb{E}(Y_{t_i} Y_{t_j} \delta X_{t_i} \delta X_{t_j} | \mathcal{F}_{t_j}) = Y_{t_i} Y_{t_j} \delta X_{t_i} \mathbb{E}(\delta X_{t_j} | \mathcal{F}_{t_j}) = 0$$

Using the law of iterated expectation we get:

$$\mathbb{E}(Y_{t_i} Y_{t_j} \delta X_{t_i} \delta X_{t_j} | \mathcal{F}_0) = \mathbb{E}(\underbrace{\mathbb{E}(Y_{t_i} Y_{t_j} \delta X_{t_i} \delta X_{t_j} | \mathcal{F}_{t_j})}_{0} | \mathcal{F}_0) = 0$$

Since the cross terms have expected value 0, we get only the first term:

$$\mathbb{E}(Z_t^2) = \lim_{\delta t \rightarrow 0} \sum_i \mathbb{E}(Y_{t_i}^2 (\delta X_{t_i})^2) = \int_0^t \mathbb{E}(Y_s^2) \sigma^2 ds.$$

We need to assume that  $Y_t \in L^2$  (“square summable”) which means that this integral is almost surely finite.

**Definition 9.3.** The *quadratic variation* of  $Z_t$  is defined to be the limit of the sum of the square increments of  $Z_t$ :

$$\langle Z \rangle_t := \lim_{\delta t \rightarrow 0} \sum (\delta Z_t)^2$$

In the example,  $\delta Z_t = Y_t \delta X_t$ . What I showed is that

$$\mathbb{E}(Z_t^2) = \mathbb{E}(\langle Z \rangle_t).$$

**9.1. Discrete stochastic integral.** I will now start from the beginning and go step by step. We know that we have to take finite sums and take a limit as  $\delta t$  goes to zero. In this section we look only at the finite sums.

9.1.0. Take discrete time:  $n = 0, 1, 2, 3, \dots$

$X_n$  is simple random walk on  $\mathbb{Z}$  with

$X_0 = 0$

$$\delta X_n = X_n - X_{n-1} = \begin{cases} 1 & \text{with probability } \frac{1}{2} \text{ "heads"} \\ -1 & \text{with probability } \frac{1}{2} \text{ "tails"} \end{cases}$$

$X_n = (\# \text{ heads} - \# \text{ tails})$  in  $n$  tosses of a fair coin.

$Y_n =$  amount that you bet on the  $n$ th toss of the coin.

$Y_n$  is  $\mathcal{F}_{n-1}$ -measurable (i.e., predictable since it is  $\mathcal{F}_t$ -measurable for all  $t < n$ ).

$$Z_n := \sum_{i=1}^n Y_i (X_i - X_{i-1}) = \text{net gain after } n \text{ tosses}$$

This is the *discrete stochastic integral*. The three key properties of this are

- (1) linearity
- (2)  $Z_n$  is a martingale
- (3)  $Z_n - \langle Z \rangle_n$  is a martingale.

9.1.1. *linearity.* This is really obvious but important. If  $a, b$  are constants and  $V_i$  is another predictable square summable process then

$$\sum_i (aY_i + bV_i) \delta X_i = a \left( \sum_i Y_i \delta X_i \right) + b \left( \sum_i V_i \delta X_i \right)$$

In other words,  $Z_n$  is a linear function of  $Y_i$ .

9.1.2.  $Z_n$  is a martingale. This is also really easy.

$$Z_{n+1} = Z_n + \underbrace{Y_{n+1}}_{\mathcal{F}_n\text{-measurable}} \delta X_{n+1}$$

$$\begin{aligned} \mathbb{E}(Z_{n+1} | \mathcal{F}_n) &= Z_n + Y_{n+1} \underbrace{\mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n)}_0 \\ &= Z_n \end{aligned}$$

So,  $Z_n$  is a martingale. In particular,

$$\mathbb{E}(Z_n) = Z_0 = 0.$$

9.1.3. *discrete quadratic variation.* We have the discrete version of the quadratic variation:

$$\langle Z \rangle_n := \sum_{i=1}^n \underbrace{(Z_i - Z_{i-1})^2}_{Y_i \delta X_i = \pm Y_i} = \sum_{i=1}^n Y_i^2$$

$$(\delta X_i = \pm 1 \Rightarrow (\delta X_i)^2 = 1.)$$

**Theorem 9.4.** *If  $\mathbb{E}(Y_i^2) < \infty$  for all  $i$  then  $Z_n - \langle Z \rangle_n$  is a martingale.*

Before I proved this, I pointed out this corollary.

**Corollary 9.5.**  $\text{Var}(Z_n) = \mathbb{E}(Z_n^2) = \sum_{i=1}^n \mathbb{E}(Y_i^2)$ .

This follows from the theorem since

$$\mathbb{E}(Z_n^2) = \mathbb{E}(\langle Z \rangle_n) = \mathbb{E}\left(\sum Y_i^2\right) = \sum \mathbb{E}(Y_i^2).$$

*Proof of theorem.*

$$\begin{aligned} Z_{n+1}^2 &= (Z_n + Y_{n+1} \delta X_{n+1})^2 \\ &= Z_n^2 + 2Z_n Y_{n+1} \delta X_{n+1} + Y_{n+1}^2 \\ \langle Z \rangle_{n+1} &= \langle Z \rangle_n + Y_{n+1}^2. \end{aligned}$$

If we subtract, the  $Y_{n+1}^2$  terms cancel and we get:

$$Z_{n+1}^2 - \langle Z \rangle_{n+1} = Z_n^2 - \langle Z \rangle_n + \underbrace{2Z_n Y_{n+1}}_{\mathcal{F}_n\text{-measurable}} \underbrace{\delta X_{n+1}}_{\mathbb{E}(-|\mathcal{F}_n)=0}$$

The cross terms have expected value zero:

$$\mathbb{E}(2Z_n Y_{n+1} \delta X_{n+1} | \mathcal{F}_n) = 2Z_n Y_{n+1} \mathbb{E}(\delta X_{n+1} | \mathcal{F}_n) = 0$$

So,

$$\mathbb{E}(Z_{n+1}^2 - \langle Z \rangle_{n+1} | \mathcal{F}_n) = Z_n^2 - \langle Z \rangle_n$$

making  $Z_n - \langle Z \rangle_n$  into a martingale.  $\square$

There is a saying about this phenomenon:

“When you take the square of a martingale, the cross terms always have vanishing expectation.”