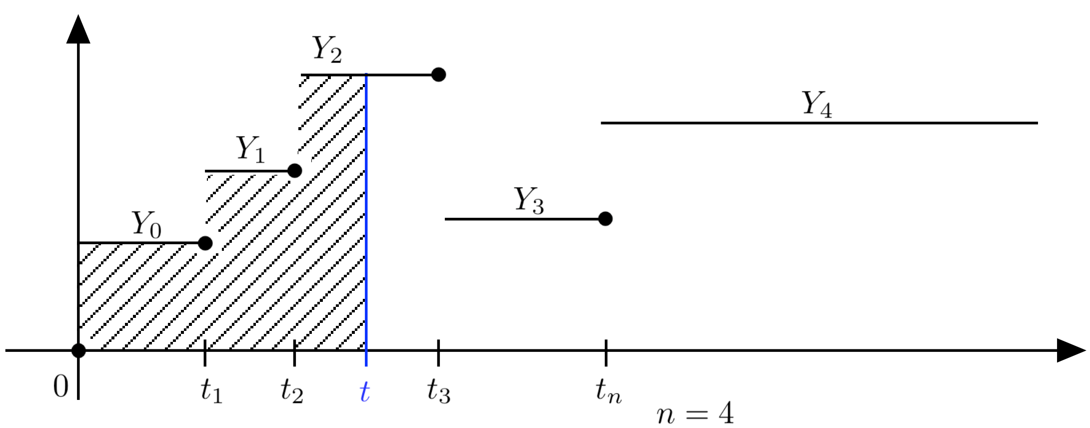
9.2. **Integration wrt Brownian motion.** We want to define the stochastic integral

$$Z_t = \int_0^t Y_s \, dW_s$$

where W_s is standard Brownian motion and Y_s is a predictable square summable process. Tomorrow, I will prove Itô's formula so we can calculate it. Today, I just want to define it. This is not that easy. I'll do it in three steps.

(1) First, we take Y_s a step function.
(2) Next, we have the theory: Z_t is a martingale and $Z_t^2 - \langle Z \rangle_t$ is also a martingale.
(3) Finally, take the limit as $\delta t \to 0$. I will use a theorem from real analysis: *The space of L^2 functions is complete.* I will explain when we get there.

9.2.1. *simple processes.* Here is the picture I drew in class.



There are only finitely many steps. The last step is infinite. A *simple process* is a step function where each step has a random height. The numbering is supposed to indicate that each step is determined at the beginning of the time interval.

$$Y_t = \begin{cases} 0 & \text{if } t = 0 \\ Y_0 & \text{if } t \in (0, t_1] \\ Y_1 & \text{if } t \in (t_1, t_2] \\ \cdots & \\ Y_n & \text{if } t \in (t_n, \infty) \end{cases}$$

We assume that Y_k is *square summable* (L^2). This means $\mathbb{E}(Y_k^2) < \infty$. (This implies that the L^2 norm of any finite part of the function, such as the shaded part, is finite.) We also assume that Y_k is $\mathcal{F}_{t_k}$. This makes Y_t into a predictable L^2 process.

Definition 9.6. The *integral* of Y_t is given by

$$\int_0^t Y_s\,ds := \sum_{i=1}^{k} Y_{i-1}(t_i - t_{i-1}) + Y_k(t - t_k) \quad \left(= \sum Y\delta t\right)$$

if $t \in (t_k, t_{k+1}]$. This integral is the shaded area in the figure.

The integral is the sum of areas of rectangles. Each rectangle has a random height. Note that at t one of these rectangles is cut in half vertically by a colored line. Each piece is again a rectangle with random height.

Definition 9.7. The *stochastic integral* of Y_t is given by

$$\int_0^t Y_s\,dW_s := \sum_{i=1}^{k} Y_{i-1}(W_{t_i} - W_{t_{i-1}}) + Y_k(W_t - W_{t_k}) \quad \left(= \sum Y\delta W\right)$$

if $t \in (t_k, t_{k+1}]$.

The first thing I pointed out is that

$$\mathbb{E}(Z_t \,|\, \mathcal{F}_0) = 0$$

This is because each term is a product $Y\delta W$ where, at some time $s \geq 0$, Y is $\mathcal{F}_s$-measurable and $\mathbb{E}(\delta W \,|\, \mathcal{F}_s) = 0$. So, using the law of iterated expectation,

$$\mathbb{E}(Y\delta W) = \mathbb{E}(\mathbb{E}(Y\delta W \,|\, \mathcal{F}_s)) = \mathbb{E}(0) = 0.$$

Since this is the third time I used this argument, I decided to make a lemma, so that, next time, I can just say "by Lemma A."

Lemma 9.8 (Lemma A)**.** *If there will be some time in the future ($\exists s \geq t$), at which you will know the value of A (A is $\mathcal{F}_s$-measurable) and B will still be random with zero expectation ($\mathbb{E}(B \,|\, \mathcal{F}_s) = 0$) then the current expected value of the product is zero:*

$$\mathbb{E}(AB \,|\, \mathcal{F}_t) = 0.$$

Theorem 9.9. *Z_t is a martingale.*

In other words, you cannot make a profit on a stock with $\mu = 0$.

Proof. All we did was to separate past from future and notice that the future stuff has expectation zero by Lemma A.

If $s > t$ then

$$Z_s = \sum_{i=1}^{m} Y_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) + Y_m\left(W_s - W_{t_m}\right)$$

$$= \sum_{i=1}^{k} Y_{i-1}\delta W + Y_k\left(W_{t_{k+1}} - W_{t_k}\right) + \text{ later terms}$$

$$\overbrace{Y_k\left(W_t - W_{t_k}\right) + Y_k\left(W_{t_{k+1}} - W_t\right)}$$

Z_t	$Z_s - Z_t$
This is in the past (before t)	This is in the future (after t)

$Z_s - Z_t$ is a sum of $Y\delta W$'s with $\mathbb{E}(Y\delta W \mid \mathcal{F}_t) = 0$ by Lemma A. So,

$$\mathbb{E}(Z_s \mid \mathcal{F}_t) = Z_t + \underbrace{\mathbb{E}(Z_s - Z_t \mid \mathcal{F}_t)}_{=0}$$

$$= Z_t.$$

So, Z_t is a martingale. □

9.2.2. *square summability.* The next question is:

$$\mathbb{E}(Z_t^2) = ?$$

If we use the shorthand: $Z_t = \sum Y\delta W$ then

$$Z_t^2 = \sum Y^2 \delta W^2 + \text{ cross terms}$$

Lemma 9.10 (Lemma B)**.** *The expected value of the cross terms is zero.*

Proof. This follows from Lemma A. In each cross terms, one part is in the past and one part is in the future. The future part is δW which has expected value 0:

$$\mathbb{E}(Y_i \delta_i W Y_j \delta_j W) = 0$$

by Lemma A since $Y_i \delta W_i Y_j$ is $\mathcal{F}_{t_j}$-measurable and $\mathbb{E}(\delta_j W \mid \mathcal{F}_{t_j}) = 0$. □

Since the cross terms don't count, you just get the square terms:

$$\mathbb{E}(Z_t^2) = \sum \mathbb{E}(Y^2 \delta W^2)$$

This is equal to:

$$= \sum \mathbb{E}(Y^2)\mathbb{E}(\delta W^2)$$

since $Y, \delta W$ are independent. In fact, we know that

$$\delta W = W_{t_i} - W_{t_{i-1}}$$

is independent of $\mathcal{F}_{t_{i-1}}$ which includes everything that happened up to time t_{i-1}. To find the expected value recall that

$$\delta W \sim N(0, \delta t).$$

So, $\mathbb{E}(\delta W^2) = \delta t$. This means that

$$\mathbb{E}(Z_t^2) = \sum \mathbb{E}(Y^2)\delta t = \int_0^t \mathbb{E}(Y_s^2) ds$$

by definition of the integral of a step function.

Definition 9.11. The *quadratic variation* of Z_t is defined by

$$\langle Z \rangle_t := \lim_{\delta t \to 0} \sum (\delta Z)^2.$$

In this case this is:

$$\langle Z \rangle_t = \lim_{\delta t \to 0} \sum (Y \delta W)^2$$

This is Z_t^2 without its cross terms. So, its expected value is:

$$\mathbb{E}(\langle Z \rangle_t) = \lim_{\delta t \to 0} \mathbb{E}\left(\sum (Y \delta W)^2\right) = \mathbb{E}(Z_t^2) = \int_0^t \mathbb{E}(Y_s^2) ds.$$

Theorem 9.12. *$Z_t^2 - \langle Z \rangle_t$ is a martingale.*

Proof. If $s > t$ then Z_s is equal to Z_t plus a sum of increments of Z:

$$Z_s = Z_t + \sum \delta Z$$

So,

$$Z_s^2 = (Z_t + \Sigma \delta Z)^2 = Z_t^2 + \sum (\delta Z)^2 + \text{cross terms}$$

And, by definition,

$$\langle Z \rangle_s = \langle Z \rangle_t + \sum (\delta Z)^2$$

If we subtract, the $\sum (\delta Z)^2$ terms cancel and we have

$$Z_s^2 - \langle Z \rangle_s = Z_t^2 - \langle Z \rangle_t + \text{cross terms}$$

But the cross terms have expectation zero:

$$\mathbb{E}(\text{cross terms} \,|\, \mathcal{F}_t) = 0$$

So,

$$\mathbb{E}(Z_2^2 - \langle Z \rangle_s \,|\, \mathcal{F}_t) = Z_t^2 - \langle Z \rangle_t$$

making $Z_t^2 - \langle Z \rangle_t$ into a martingale. ∎

9.2.3. *general stochastic integral.* Suppose now that Y_t is any L^2 predictable process. Then, we make it into a simple process by dividing the line $[0,t]$ into n pieces at points $t_k = kt/n$ and letting

$$Y_k^{(n)} = \text{average of } Y_t \text{ on } (t_{k-1}, t_k]$$
$$= \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} Y_t \, dt$$

Then you define:

$$Z_t^{(n)} = \int_0^t Y_s^{(n)} dW_s.$$

Since $Y_s^{(n)}$ is a predictable L^2 process, $Z_t^{(n)}$ is a square summable martingale.

Definition 9.13. The *general stochastic integral* $Z_t = \int_0^t Y_t dW_s$ is defined to be the limit:

$$Z_t = \lim_{n\to\infty} Z_n^{(n)}.$$

This limit exists and is L^2 by the following theorem.

Theorem 9.14. *The space of L^2 functions is complete in the L^2 norm.*

The word *complete* means that any Cauchy sequence converges in the L^2 norm. So, the theorem is saying that, if we know that $Z_t^{(n)}$ is a Cauchy sequence, we know it will converge.

Lemma 9.15. $\{Z_t^{(n)}\}$ *is a Cauchy sequence. In other words,* $\forall \epsilon > 0$ $\exists N$ *so that*

$$\mathbb{E}((Z_t^{(n)} - Z_t^{(m)})^2) < \epsilon$$

if $n, m > N$.

Remark 9.16. The usual definition of the L^2 norm is the integral of the square of a function with respect to the given measure:

$$||g||_2 := \int g^2 \, d\mu$$

But, we have a probability space and our measure is the probability measure $d\mu = f(x)dx$. So,

$$||g||_2 = \mathbb{E}(g^2).$$

I ran of out time. But there were a couple more things:

Theorem 9.17. Z_t *is a martingale.*

Proof. This follows from the fact that $\mathbb{E}(-\,|\,\mathcal{F}_s)$ is continuous in the L^2-norm. In other words, it commutes with limits. So

$$\mathbb{E}(Z_t\,|\,\mathcal{F}_s) = \mathbb{E}\left(\lim_{n\to\infty} Z_t^{(n)}\,|\,\mathcal{F}_s\right) = \lim \mathbb{E}(Z_t^{(n)}\,|\,\mathcal{F}_s)$$
$$= \lim Z_s^{(n)} = Z_s$$

making Z_t a martingale. □

Since the cross terms for the product of a sum of increments of a martingale have expectation zero (by Lemma A), we have:

Corollary 9.18. $Z_t^2 - \langle Z\rangle_t$ *is a martingale.*