

9.4. **Extensions of Itô's formula.** First I explained the idea of “covariation” and the product rule. Then I used this to derive the second and third version of Itô's formula.

9.4.1. *covariation.*

Definition 9.25. The *covariation process* of A_t and B_t is defined by

$$\langle A, B \rangle_t := \lim_{\delta t \rightarrow 0} \sum \delta A \delta B = \lim_{\delta t \rightarrow 0} \sum (A_{t_i} - A_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})$$

I used the word “process” to emphasize the fact that this is random and it varies with time.

Mathematically, this is the symmetric bilinear form which is associated to the quadratic variation. Two properties follow immediately from the definition:

Properties

$$\langle A, A \rangle_t = \langle A \rangle_t \quad (\text{quadratic variation})$$

$$d \langle A, B \rangle_t = dA_t dB_t \quad (\text{by definition})$$

And we have another formula which can also be used to define $\langle A, B \rangle_t$ (since the other three terms have already been defined):

$$\langle A + B \rangle_t = \langle A \rangle_t + \langle B \rangle_t + 2 \langle A, B \rangle_t.$$

This is “obvious” from the binomial formula:

$$\sum (\delta_i A + \delta_i B)^2 = \sum (\delta_i A)^2 + \sum (\delta_i B)^2 + 2 \sum \delta_i A \delta_i B.$$

9.4.2. *product rule.* One of the really nice uses of the new definition is the following product formula which holds without error! (See picture.)

$$\delta(AB) = A\delta B + B\delta A + \delta A \delta B$$

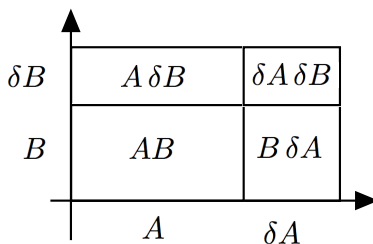


FIGURE 1. The term $\delta A \delta B$ becomes the covariation.

The infinitesimal version is

$$d(AB) = AdB + BdA + d\langle A, B \rangle_t.$$

9.4.3. *bounded variation.* Last time (Exercise 9.23) I showed that:

Lemma 9.26 (Lemma 1). *If f is continuous with bounded variation (e.g. if f is C^1) then*

$$\langle f \rangle_t = 0$$

(The quadratic variation of f is zero.)

Using the same argument as in the proof of the Schwarz inequality, you get the following.

Lemma 9.27 (Lemma 2). *If $\langle f \rangle_t = 0$ and $\langle X \rangle_t < \infty$ then $\langle f, X \rangle_t = 0$ a.s. for all t .*

Proof. Suppose that, at some time t , there is a chance (nonzero probability) that $\langle f, X \rangle_t \neq 0$. Say, it could be positive. Then there is a possibility that $\langle f, X \rangle_t > c > 0$. Then

$$\langle X - af \rangle_t = \langle X \rangle_t - 2a \underbrace{\langle f, X \rangle_t}_{>c} + a^2 \underbrace{\langle f \rangle_t}_0$$

If we make a really big then we get $\langle X - af \rangle_t < 0$ with nonzero probability. But this is impossible because quadratic variations are sums of squares! This contradiction proves the lemma. \square

9.4.4. *review.* During the review I used the notation

$$d\langle A, B \rangle_t = dA_t dB_t.$$

Then the properties of covariation become:

- (1) (Lévy) $d\langle W \rangle_t = (dW_t)^2 = dt$.
- (2) $d\langle f, X \rangle_t = df dX_t = 0$ if f has bdd variation.
- (3) (product rule) $d(AB) = AdB + BdA + dA dB$.

Before doing Itô's second and third formulas, I went over Itô's first formula which was:

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

Exercise 9.28. Take $f(x) = e^{\sigma x}$. Then $f'(x) = \sigma e^{\sigma x}$, $f''(x) = \sigma^2 e^{\sigma x}$. So, the formula gives:

$$de^{\sigma x} = \sigma e^{\sigma x} dW_t + \frac{1}{2}\sigma^2 e^{\sigma x} dt$$

$$de^{\sigma x} = e^{\sigma x} \left(\sigma dW_t + \frac{\sigma^2}{2} dt \right).$$

This is very similar to the model for stock prices that we will be using:

$$S_t := \text{value of one share of stock at time } t.$$

Our model assumes that

$$\boxed{dS_t = S_t(\sigma dW_t + \mu dt)}$$

where

$$\left. \begin{array}{l} \mu = \text{drift} \\ \sigma = \text{volatility} \end{array} \right\} \text{assumed constant}$$

We are not assuming that $\mu = \sigma^2/2$. However, it would be convenient if this were true.

9.4.5. *quadratic variation of Z_t .* Here I calculated the quadratic variation of Z_t .

Theorem 9.29. *Suppose that*

- (1) $dZ_t = X_t dW_t + Y_t dt$ where
- (2) Y_t is integrable (L^1), i.e., $\int_0^t |Y_s| ds < \infty$ and
- (3) X_t is square summable (L^2), i.e., $\int_0^t X_s^2 ds < \infty$.

Then

$$d\langle Z \rangle_t = X_t^2 dt.$$

First, I rephrased the statement: Condition (1) says that

$$Z_t - Z_0 = \underbrace{\int_0^t X_s dW_s}_{=B_t \text{ stochastic integral}} + \underbrace{\int_0^t Y_s ds}_{f(t)}$$

The conclusion is:

$$\langle Z \rangle_t = B_t + f(t).$$

Proof. $f(t)$ has bounded variation since

$$\text{variation of } f = \int_0^t |f'(s)| ds = \int_0^t |Y_s| ds < \infty$$

since Y_t is L^1 . But then

$$Z_t = B_t + \underbrace{f(t) + Z_0}_{g(t)} = B_t + g(t)$$

where $g(t)$ has bounded variation because it has the same derivative as $f(t)$. So,

$$\begin{aligned} \langle Z \rangle_t &= \langle B \rangle_t + \underbrace{\langle g \rangle_t}_{=0 \text{ by Lem 1}} + 2 \underbrace{\langle B_t, g \rangle_t}_{=0 \text{ by Lem 2}} = \langle B \rangle_t \\ &= dB_t dB_t = X_t^2 (dW_t)^2 = X_t^2 dt. \end{aligned}$$

□

9.4.6. *Itô's second formula.* We want to replace W_t with

$$Z_t = \int_0^t X_s dW_s + \int_0^t Y_s ds$$

in Itô's first formula. The question is: What is

$$df(Z_t) = ?$$

Taylor's formula says:

$$\delta f = f(x + \delta x) - f(x) = f'(x)\delta x + \frac{1}{2}f''(x)(\delta x)^2 + O((\delta x)^3)$$

Substituting $x = Z_t$ we get:

$$\delta f(Z_t) = f(Z_{t+\delta t}) - f(Z_t) = f'(Z_t)\delta Z_t + \frac{1}{2}f''(Z_t)(\delta Z_t)^2 + \epsilon$$

Taking the limit as $\delta t \rightarrow 0$ we get:

$$df(Z_t) = f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)d\langle Z \rangle_t$$

Now we can insert expressions for dZ_t and $d\langle Z \rangle_t$:

$$dZ_t = X_t dW_t + Y_t dt$$

$$d\langle Z \rangle_t = X_t^2 dt$$

and we get:

Theorem 9.30 (Itô II).

$$df(Z_t) = f'(Z_t)X_t dW_t + f'(Z_t)Y_t dt + \frac{1}{2}f''(Z_t)X_t^2 dt.$$

Exercise 9.31. Calculate $d(S_t^2)$ if $\mu = -5$ and $\sigma = 3$. [Use $f(x) = x^2$.]

Answer: $f(x) = x^2$. So, $f'(x) = 2x$, $f''(x) = 2$ and Itô's equation gives:

$$dS_t^2 = 2S_t X_t dW_t + 2S_t Y_t dt + X_t^2 dt.$$

The formula for dS_t is:

$$dS_t = \underbrace{\sigma S_t}_{X_t} dW_t + \underbrace{\mu S_t}_{Y_t} dt$$

So, $X_t = 3S_t$ and $Y_t = -5S_t$ making

$$dS_t^2 = 6S_t^2 dW_t - 10S_t^2 dt + 9S_t^2 dt = 6S_t^2 dW_t - S_t^2 dt.$$

9.4.7. *Itô's third formula.* We need the next formula which gives the differential of a function of t and Z_t :

$$df(t, Z_t) = ?$$

The function we really want to know about is

$$V(t, x) = \text{the value at time } t \text{ of a stock option if } S_t = x$$

The kind of stock option we are talking about is the option to buy one share of stock at some future time T at a fixed price K . We want to know how much the option is worth:

$$V(t, S_t) = ?$$

A general function of this kind is $f(t, Z_t)$. This is a function of the vector

$$\mathbf{x} = \begin{pmatrix} t \\ Z_t \end{pmatrix}, \quad \delta \mathbf{x} = \begin{pmatrix} \delta t \\ \delta Z_t \end{pmatrix}$$

The derivative of f is given by the *gradient*

$$\nabla f = (\dot{f}, f') = \left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial z} \right)$$

Here I use the physicist notation that a dot indicates time derivative and prime indicates space derivative:

$$\dot{f} = \frac{\partial f}{\partial t}, \quad f' = \frac{\partial f}{\partial z}$$

The second derivative of f is given by the *Hessian* of f :

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial t^2} & \frac{\partial^2 f}{\partial t \partial z} \\ \frac{\partial^2 f}{\partial t \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \ddot{f} & \dot{f}' \\ \dot{f}' & f'' \end{pmatrix}$$

This is a symmetric matrix which tells you if you have a local max or min or a saddle point (from multivariable calculus). Taylor's formula in two variables is:

$$\delta f = \nabla f \cdot \delta \mathbf{x} + \frac{1}{2} D^2 f (\delta \mathbf{x})^2 + \epsilon$$

The first term on the right is the dot product of vectors (which is a matrix product when ∇f is written as a row vector):

$$\nabla f \cdot \delta \mathbf{x} = (f, f') \begin{pmatrix} \delta t \\ \delta Z_t \end{pmatrix} = \dot{f} \delta t + f' \delta Z_t$$

The second term is a product of three matrices:

$$\frac{1}{2} D^2 f (\delta \mathbf{x})^2 = \frac{1}{2} (\delta t, \delta Z_t) \begin{pmatrix} \ddot{f} & \dot{f}' \\ \dot{f}' & f'' \end{pmatrix} \begin{pmatrix} \delta t \\ \delta Z_t \end{pmatrix}$$

So,

$$\delta f = \dot{f} \delta t + f' \delta Z_t + \frac{1}{2} (\delta t, \delta Z_t) \begin{pmatrix} \ddot{f} & \dot{f}' \\ \dot{f}' & f'' \end{pmatrix} \begin{pmatrix} \delta t \\ \delta Z_t \end{pmatrix} + \epsilon$$

Take the limit as $\delta t \rightarrow 0$ and plug in $d\langle t \rangle = 0$, $d\langle t, Z_t \rangle = 0$ and $(dZ_t)^2 = X_t^2 dt$:

$$df(t, Z_t) = \dot{f} dt + f' dZ_t + \frac{1}{2} f'' X_t^2 dt$$

Since $dZ_t = X_t dW_t + Y_t dt$ we get:

Theorem 9.32 (Itô III).

$$df(t, Z_t) = \dot{f} dt + f' X_t dW_t + f' Y_t dt + \frac{1}{2} f'' X_t^2 dt.$$

Exercise 9.33. If a, b, c are constants then find dZ_t when

$$Z_t = at^2 + btW_t + cW_t^2$$

[Use $f(t, x) = at^2 + btx + cx^2$]

Answer: We need:

$$\dot{f} = 2at + bx = 2at + bW_t$$

$$f' = bt + 2cx = bt + 2cW_t$$

$$f'' = 2c$$

So,

$$\begin{aligned} df(t, W_t) &= (2at + bW_t)dt \\ &+ (bt + 2cW_t)(X_t dW_t + Y_t dt) \\ &+ \frac{1}{2}(2c)X_t^2 dt \end{aligned}$$

Since we are using $dW_t = 1dW_t + 0$ instead of $dZ_t = X_t dW_t + Y_t dt$ in Itô's equation, we need to put $X_t = 1$ and $Y_t = 0$. This gives:

$$df(t, W_t) = (bt + 2cW_t)dW_t + (2at + bW_t + c)dt$$

We can also do this using just the product formula:

$$d(at^2) = 2atdt$$

$$d(btW_t) = bW_t dt + bt dW_t$$

$$d(cW_t^2) = 2cW_t dW_t + cdt$$

Adding these together, we get the same answer as before.