

9.5. Black-Scholes Equation. On the last day of class I derived the Black-Scholes equation and solved it in the case when the drift and volatility of the stock is constant.

9.5.1. *the question.* First I explained the problem.

S_t = value of one share of stock at time t .

We have an option to buy one share of stock for $\$K$ (the *exercise price*) at time T (the *expiry date*). The question is:

How much is this option worth at time t if $S_t = x$?

Call the value of the option V . So:

$V(t, x)$ = value of the option at time t if $S_t = x$.

We want an equation with a variable x . This way, we don't need the actual price of the stock. This is hard to calculate. But, we know what the value will be at expiry ($t = T$):

a) If $S_T < K$ then this option is worth $V = 0$. If, e.g., the stock is worth $\$90$, the option to buy it for $\$100$ is worthless.

b) If $S_T \geq K$ then $V = S_T - K$: You should exercise the option to buy the stock for K and sell it for S_T making a profit of $S_T - K$. For example, if the stock is worth $\$110$ and you have an option to buy it for $\$100$, you should exercise this option, buy the stock and then sell it making a profit of $\$10$.

So,

$$V(T, S_T) = \begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

We write this as $(S_T - K)_+$. We can also write this as:

$$V(T, x) = (x - K)_+$$

9.5.2. *replicating portfolio.* To calculate $V(t, x)$ for any $t \leq T$ we construct a portfolio of stocks and bonds which is equivalent to the option. Since we can buy these stocks and bonds today, we know the current value of the portfolio. This will be the value of the option. This combination of stocks and bonds is called a *replicating portfolio*. We will start with the assumption that such a portfolio exists. Then we get an equation which we can solve. Halfway through the calculation we will argue that the solution of the equation is in fact a replicating portfolio.

A portfolio consists of stocks and bonds. We have only one stock and one bond. One share of stock is worth S_t . Bonds increase at a certain fixed rate r . If the bond has a yield of 6% at the end of one

year, then

$$r = \ln(1.06) = .05826891$$

The rate $r = 5.8\%$ is the exponential rate. If this interest rate is compounded continuously, we would get 6% at the end of one year. In general, if you have $\$Y$ in bonds, its value after δt years is

$$e^{r\delta t}Y.$$

The value of a portfolio is

$$O_t = X_t S_t + Y_t$$

where

$$X_t = \# \text{ shares of stock}$$

$$Y_t = \$ \text{ in bonds.}$$

If you want your portfolio O_t to imitate your stock option $V(t, S_t)$, you need O_t to be *self-financing*. This means you don't put money in or take it out. But you can sell stock to buy bonds and vice versa.

The next question is: What is

$$dO_t = ?$$

There are two answers:

(a)

$$\begin{aligned} dO_t &= \boxed{X_t dS_t} + rY_t dt \\ &= X_t \mu S_t dt + \underbrace{X_t \sigma S_t dW_t}_{\text{stochastic part}} + rY_t dt \end{aligned}$$

Each share of stock increases in value by dS_t , so $X_t dS_t$ is the increase in value of your portfolio as a result of the stock. Your bonds increase in value at a constant rate and gives $r dt$ for each dollar in bonds.

(b) We can also use the product rule to calculate dO_t :

$$dO_t = (dX_t)S_t + \boxed{X_t dS_t} + d\langle X, S \rangle_t + dY_t.$$

O_t is self-financing if and only if (a) = (b).

There is another formula for dO_t . Since O_t is a replicating portfolio we have $O_t = V(t, S_t)$.

(c) Using Itô III we have:

$$dO_t = dV = \dot{V} dt + \boxed{V' dS_t} + \frac{1}{2} V'' \underbrace{d\langle S \rangle_t}_{\sigma^2 S_t^2 dt}$$

This is because $dS_t = \mu S_t dt + \sigma S_t dW_t$.

If we expand dS_t in (c), we get:

$$dO_t = dV = \dot{V}dt + V'\mu S_t dt + \underbrace{V'\sigma S_t dW_t}_{\text{stochastic part}} + \frac{1}{2}V''\sigma^2 S_t^2 dt$$

If we compare this with (a) we see that there is only one stochastic term in each. These must be equal.

$$\Rightarrow \boxed{1} \quad X_t = V'(t, S_t)$$

Interpretation: This means that $X_t = V'$ shares of stock has the same volatility as the option. So, if we take the difference, you have a financial instrument with zero risk. Now you can argue that this must be increasing at the bond rate since, otherwise, there would be an opportunity for “arbitrage” (someone could buy and/or sell this to make money). So,

$$V(t, S_t) - X_t S_t = Y_t.$$

Since $X_t = V'$, the two boxed terms in (a) and (c) are equal. The other three terms give:

$$\Rightarrow \boxed{2} \quad rY_t = \dot{V} + \frac{\sigma^2}{2}V''S_t^2$$

But, $Y_t = O_t - X_t S_t$. So,

$$rY_t = rO_t - rX_t S_t = rV - rV' S_t.$$

Comparing this with $\boxed{2}$ and putting $S_t = x$ we get the *Black-Scholes Equation*:

$$\boxed{\dot{V}(t, x) + \frac{\sigma^2}{2}V''(t, x)x^2 - rV(t, x) + rxV'(t, x) = 0.}$$

Note that the only information that we need is the volatility σ and the bond rate r .

9.5.3. *simplification.* Since μ does not occur in the equation, we can assume that $\mu = 0$! So,

$$dS_t = \sigma S_t dW_t$$

This means that S_t is a martingale.

Next, we can assume that $r = 0$!! This is because we can think of r as the *inflation rate* and we can correct for it later. The inflation rate tells us how much one dollar today (at time t) will be worth at expiry (time T):

$$x\$ \text{ today } t = e^{r\delta t} x \text{ in the future } T$$

The value of the stock $x = S_t$ is in today's dollars. The option to buy for K is in tomorrow's dollars:

$$\begin{array}{ccc} e^{-r\delta t} K \$ & = & \$K \\ \text{at time } t & & \text{at expiry } T \end{array}$$

Here $\delta t = T - t$ is the time until expiry.

I calculated the value of the option assuming $r = 0$ and called it $V_0(t, x)$. If $r \neq 0$, we have to adjust the answer for inflation by extrapolating the present price of the stock x to the future, calculating the value in terms of tomorrow's dollars (since K is in tomorrow's dollars) and then extrapolating back to today:

$$V(t, x) = e^{-r\delta t} V_0(t, e^{r\delta t} x).$$

9.5.4. *solution of simplified Black-Scholes.* If $\mu = r = 0$ then

$$dS_t = \sigma S_t dW_t$$

$$dV_0 = dO_t = X_t dS_t = X_t \sigma S_t dW_t$$

$\Rightarrow S_t, V_0, O_t$ are all martingales. (E.g., $\mathbb{E}(dS_t) = \sigma \mathbb{E}(S_t) \mathbb{E}(dW_t) = 0$.)

This implies that

$$V_0(t, S_t) = \mathbb{E}(\underbrace{V_0(T, S_T)}_{(S_T - K)_+} | \mathcal{F}_t)$$

Next, we need the formula for S_T .

Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. So,

$$\begin{aligned} d \ln S_t &= f'(S_t) dS_t + \frac{1}{2} f''(S_t) (\sigma S_t)^2 dt \\ &= \frac{dS_t}{S_t} - \frac{1}{2} \frac{1}{S_t^2} \sigma^2 S_t^2 dt \\ &= \sigma dW_t - \frac{\sigma^2}{2} dt \\ &= d \left(\sigma W_t - \frac{\sigma^2 t}{2} \right) \end{aligned}$$

So,

$$\ln S_t = \sigma W_t - \frac{\sigma^2 t}{2} + \ln S_0$$

Exponentiating, we get:

$$S_t = S_0 e^{\sigma W_t - \sigma^2 t/2}$$

Since today is time t (not 0) and the future is time T (not t), this should be written as:

$$S_T = S_t e^{\sigma W_{\delta t} - \sigma^2 \delta t/2}$$

where $\delta t = T - t$. Plugging this into the formula for V_0 we get:

$$V_0(t, S_t) = \mathbb{E}((S_t e^{\sigma W_{\delta t} - \sigma^2 \delta t / 2} - K)_+)$$

We can calculate this expected value:

Theorem 9.34.

$$V_0(t, x) = x \Phi\left(\frac{\ln(x/K) + \sigma^2 \delta t / 2}{\sigma \sqrt{\delta t}}\right) - K \Phi\left(\frac{\ln(x/K) - \sigma^2 \delta t / 2}{\sigma \sqrt{\delta t}}\right)$$

where $\Phi(z)$ is the cdf for the standard normal distribution:

$$\Phi(z) = \int_{-\infty}^z dy \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

Proof. At this point, I started to use the abbreviation:

$$a = \sigma \sqrt{\delta t}$$

and, as before, $x = S_t$. I pointed out that

$$\sigma W_{\delta t} \sim N(0, \underbrace{\sigma^2 \delta t}_{a^2})$$

This means that,

$$z = \frac{\sigma W_{\delta t}}{\sigma \sqrt{\delta t}} = \frac{\sigma W_{\delta t}}{a} \sim N(0, 1)$$

and

$$\sigma W_t = az.$$

Then the formula for V_0 is:

$$V_0(t, x) = \mathbb{E}((x e^{az - a^2/2} - K)_+)$$

Next, we need to know when is $x e^{az - a^2/2} - K \geq 0$. This is the same as:

$$\begin{aligned} x e^{-a^2/2} e^{az} &\geq K \\ e^{az} &\geq \frac{K}{x} e^{a^2/2} \\ az &\geq \ln\left(\frac{K}{x}\right) + \frac{a^2}{2} \\ z &\geq \frac{1}{a} \ln\left(\frac{K}{x}\right) + \frac{a}{2} \end{aligned}$$

Since z is standard normal,

$$\mathbb{E}((x e^{az - a^2/2} - K)_+) = \int_{\frac{1}{a} \ln(\frac{K}{x}) + \frac{a}{2}}^{\infty} dz \left(x e^{az - a^2/2} - K \right) \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

This is actually very easy to integrate. You just put

$$z = y + a, \quad dz = dy.$$

Then

$$y = z - a \geq \frac{1}{a} \ln \left(\frac{K}{x} \right) - \frac{a}{2}$$

So, $\mathbb{E}((xe^{az-a^2/2} - K)_+)$

$$\begin{aligned} &= \int_{\frac{1}{a} \ln(\frac{K}{x}) - \frac{a}{2}}^{\infty} dy x \frac{e^{-y^2/2}}{\sqrt{2\pi}} - \int_{\frac{1}{a} \ln(\frac{K}{x}) + \frac{a}{2}}^{\infty} dz K \frac{e^{-z^2/2}}{\sqrt{2\pi}} \\ &= x \left(1 - \Phi \left(\frac{1}{a} \ln \left(\frac{K}{x} \right) - \frac{a}{2} \right) \right) - K \left(1 - \Phi \left(\frac{1}{a} \ln \left(\frac{K}{x} \right) + \frac{a}{2} \right) \right) \end{aligned}$$

But

$$1 - \Phi(z) = \Phi(-z).$$

So, this can be simplified to:

$$= x \Phi \left(\frac{1}{a} \ln \left(\frac{x}{K} \right) + \frac{a}{2} \right) - K \Phi \left(\frac{1}{a} \ln \left(\frac{x}{K} \right) - \frac{a}{2} \right)$$

which proves the theorem when you insert $a = \sigma\sqrt{\delta t}$. \square

When you adjust this for inflation you get the answer for $r \neq 0$:

Corollary 9.35.

$$V(t, x) = e^{-r\delta t} V_0(t, e^{r\delta t} x)$$

$$= x \Phi \left(\frac{\ln(x/K) + \sigma^2 \delta t / 2 + r \delta t}{\sigma \sqrt{\delta t}} \right) - e^{-r\delta t} K \Phi \left(\frac{\ln(x/K) - \sigma^2 \delta t / 2 + r \delta t}{\sigma \sqrt{\delta t}} \right)$$