

MATH 56A: INTRODUCTION TO STOCHASTIC PROCESSES AND  
MODELS

A *stochastic process* is a random process which evolves with time. The basic model is the *Markov chain*. This is a set of “states” together with transition probabilities from one state to another. For example, in simple epidemic models there are only two states:  $S$  = “susceptible” and  $I$  = “infected.” The probability of going from  $S$  to  $I$  increases with the size of  $I$ . In the simplest model The  $S \rightarrow I$  probability is proportional to  $I$ , the  $I \rightarrow S$  probability is constant and time is discrete (for example, events happen only once per day). In the corresponding deterministic model we would have a “difference equation.”

In a continuous time Markov chain, transition events can occur at any time with a certain probability density. The corresponding deterministic model is a first order differential equation. This includes the “general stochastic epidemic.”

The number of states in a Markov chain is either finite or countably infinite. When the collection of states becomes a continuum, e.g., the price of a stock option, we no longer have a “Markov chain.” We have a more general stochastic process. Under very general conditions we obtain a Wiener process, also known as Brownian motion. The mathematics of hedging implies that stock options should be priced as if they are exactly given by this process. Ito’s formula explains how to calculate (or try to calculate) stochastic integrals which give the long term expected values for a Wiener process.

This course will be a theoretical mathematics course. The goal is to give rigorous mathematical definitions and derive the consequences of simple stochastic models, most of which will be Markov chains. I will not explain the principles of biology, economics and physics behind the models, although I would invite more qualified guest lecturers to explain the background for the models. There will be many examples, not just the ones outlined above.

The prerequisite for the course will be Math 36a (probability using calculus), linear algebra (Math 15) and multivariable calculus (Math 20). Basic linear differential equations will be reviewed at the beginning of the course. Probability will not be reviewed. This is an advanced course for students already familiar with probability. Linear algebra is also heavily used. Statistics is not required. However, knowledge of Statistics can substitute for knowledge of Probability.

## OUTLINE OF COURSE

- (1) Review of linear differential and difference equations.
- (2) Definitions and general notions about stochastic processes
- (3) Finite and countably infinite Markov chains
- (4) Continuous time Markov chains
- (5) Renewal processes
- (6) Martingales
- (7) Wiener processes (Brownian motion)
- (8) Stochastic integration and Ito's formula
- (9) Applications to population biology and epidemiology
- (10) Application to financial security analysis

Applications will vary according to the interests of students and teacher.

**Required text** *Introduction to Stochastic Processes, 2nd ed.*, Gregory Lawler, Chapman & Hall

Recommended books:

- *Markov Chains*, J.R. Norris, Cambridge University Press. (This is an excellent book which develops Markov chains in a more leisurely way but does not have stochastic integrals.)
- *Epidemic Modelling, An Introduction*, D.J. Daley & J.Gani, Cambridge University Press
- *Financial Derivatives, Pricing, Applications and Mathematics*, Jamil Baz & George Chacko, Cambridge University Press
- *The Mathematics of Financial Derivatives, A Student Introduction*, Paul Wilmott, Sam Howison, Jeff Dewynne, Cambridge University Press

**Grading** 3/7 homework, 2/7 quizzes, 1/7 in-class participation, 1/7 take home exam. Expected grade: A-/B+

**Homework** There will be weekly homework. The first HW might have the problem: Using the SIR model, prove that the number of infected reaches its highest point when the size of the susceptible population reaches the threshold. Students are encouraged to work on their homework in groups and to access all forms of aid including expert advice, internet and other resources. The work you hand in should, however, be in your own words and in your own handwriting. And you should understand what you have written. Every student hands in his own homework.

**Quizzes** will be given 2 or 3 weeks. Students registered in the class should form groups of 2, 3 or 4 to work on these problems in class, solve them and help other students in the group to understand them.

Students auditing the class can also participate in a way that I will discuss in class. Each group should hand in their answers signed by all members of the group. There is a special rule for Quiz 2. You cannot form the same groups as in Quiz 1. You must form a new group for Quiz 2. Afterwards there are no rules, except that the group is limited to at most 4.

**Take Home Exam** Has the same rules as Quizzes (except you do it at home), i.e., students should form groups of 2,3 or 4 and each group hands in one report. Also: Each of you needs to evaluate the others in your group. If I gave you \$100 for the job, how would you split the fee with your partners? (Normally you split the fee evenly. But sometimes this is just not right.)

**Attendance** is required (Don't miss two classes in a row!) and counts as part of the grade.

**Students with disability** If you are a student with a documented disability at Brandeis University and if you wish to request a reasonable accommodation for this class, please see the instructor immediately.

**Academic integrity** All members of the Brandeis academic community are expected to maintain the highest standard of academic integrity as outlined in "Rights and Responsibilities." Any violations will be treated seriously.