MATH 56A: STOCHASTIC PROCESSES
QUIZ

From the syllabus: Quizzes will be given [every] 2 or 3 weeks. Students registered in the class should form groups of 2, 3 or 4 to work on these problems in class, solve them and help other students in the group to understand them. Students auditing the class can also participate in a way that I will discuss in class. Each group should hand in their answers signed by all members of the group. There is a special rule for Quiz 2. You cannot form the same groups as in Quiz 1. You must form a new group for Quiz 2. Afterwards there are no rules, except that the group is limited to at most 4.

Math 56a, Practice Quiz I

Here are my answers. If I have comments after grading the practice quiz, I will post another blurb.

1. Convert the following second order difference equation into a first order matrix equation (including the initial conditions). Don't solve it.

$$f(n) = 2f(n-1) + 3f(n-2), \quad f(0) = 5, f(1) = 6.$$

First let

$$g(n) := f(n-1).$$

Then the original equation becomes:

$$f(n) = 2f(n-1) + 3g(n-1)$$

In matrix form these two equations become:

$$\begin{pmatrix} f(n) \\ g(n) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(n-1) \\ g(n-1) \end{pmatrix}$$

Which you can also write as:

$$X_n = AX_{n-1}, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

The initial condition is:

$$X_0 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

2. Find the invariant probability distribution for

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

The answer is the normalized left null vector of $P - I$ (or $I - P$):

$$P - I = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & 1/4 & -1/4 \end{pmatrix}$$

This calculation is easy. You don't need to column reduce the matrix. But, in case you did, you should have gotten:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 0 \end{pmatrix}$$

with null vector $(0, 1, 2)$ which normalizes to

$$\pi = (0, 1/3, 2/3).$$

Since there is only one recurrent class $R_1 = \{2, 3\}$, there is only one invariant distribution. (The invariant distribution π is unique.)

3. Make a Markov chain model for the following problem.

Every morning you get up and think about loosing weight. You want to skip some meals. Half the time you skip breakfast and go to class hungry. Half the time you get breakfast. When you go to breakfast, you often (with probability 0.6 meet someone and make plans for dinner). You always skip lunch. If you skipped breakfast, you always go for dinner. If you ate breakfast and didn't make plans for dinner, then you skip dinner. If you made plans with someone, then you don't forget and you go for dinner. If you eat twice, then the next day, you skip breakfast. Otherwise, you follow the same plan the next day.

What are the states? What is the transition matrix? What is the initial probability distribution?

Here are three answers:

Answer 1 is based on your state at the beginning of your meals:

You have 5 states: B_0, B_1, D_0, D_1, D_2:

B_0 means you skipped breakfast.

B_1 means you go to eat breakfast.

D_0 means you skip dinner.

D_1: You eat dinner alone.

D_2: You eat dinner with a friend.

The transition matrix is:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the initial distribution is

$$\alpha = (1/2, 1/2, 0, 0, 0)$$

Answer 2 is based on your state at the end of your meal:

B_0: You skipped breakfast.

B_1: You ate breakfast but did not make a dinner date.

B_2: You ate breakfast and made plans for dinner.

D_0: You skipped dinner.
D_1: You ate dinner but that was your only meal.
D_2: You ate dinner and that was your second meal.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ .5 & .2 & .3 & 0 & 0 & 0 \\ .5 & .2 & .3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = (.5, .2, .3, 0, 0, 0).$$

Answer 3: Think about the whole day. There are only 3 possibilities:
D: You skipped breakfast and ate dinner.
B: You ate breakfast and skipped dinner.
A: You ate breakfast and dinner.

$$P = \begin{pmatrix} .5 & .2 & .3 \\ .5 & .2 & .3 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\alpha = (.5, .2, .3).$$

My opinion is that Answer 1 is the best (among these three). The reason is that the entries in the matrix P and α contain the raw data. Answers 2 and 3 are partially processed data since they contain the numbers $(1/2)(0.4) = 0.2$ and $(1/2)(0.6) = 0.3$ indicating that two transitions have been compressed into one transition.

I think it is better to avoid calculations in the word-to-equation process: Leave the calculations to your computer!