**Problem 1** Consider the countable Markov chain with state space
\[ S = \{0, 1, 2, 3, \ldots \} \]
and probabilities \( p(n, n+1) = p, p(n, 0) = q = 1 - p. \)

For which values of \( p \) is this chain positive recurrent, null recurrent, transient?

The first point you need to understand is: This chain is transient if \( p = 1 \) because, in that case, you keep increasing with probability one. The chain is not even irreducible.

If \( p < 1 \) then there is always a positive probability \( q \) of going to 0. So, you keep coming to 0 with probability one. So, \( p < 1 \) implies recurrent.

\( X_n \) is positive recurrent if it has an invariant distribution \( \pi \). This is a probability vector satisfying:
\[ \pi(n) = \sum \pi(m)p(m, n) \]
If \( n > 0 \) there is only one term in the sum:
\[ \pi(n) = \pi(n-1)p = \pi(n-2)p^2 = \cdots = \pi(0)p^n \]
If \( n = 0 \) then
\[ \pi(0) = \sum_{n=0}^{\infty} \pi(n)q = (1)q = q \]
So,
\[ \pi(n) = qp^n \]
The sum of these numbers is
\[ \sum \pi(n) = \sum_{n=0}^{\infty} qp^n = \frac{q}{1-p} = \frac{1}{1} = 1 \]
So, everything checks out and we see that \( X_n \) is positive recurrent when \( p < 1 \). It is transient if \( p = 1 \) and it is never null recurrent.

**Problem 2** Consider the continuous Markov chain \( X_t \) with states
\[ S = \{0, 1, 2, 3, \ldots \} \]
and transition rates given by \( \alpha(n, n+1) = 2, \alpha(n, n-2) = 2. \)

(a) Convert this to a countable chain \( Z_n. \)

This is very easy. Since the rates are equal we have:
\[ p(n, n+1) = 1/2, \quad p(n, n-2) = 1/2. \]

(b) Show that \( X_t \) is xxx positive recurrent. (Oops, it is positive recurrent.)

\( X_t \) is positive recurrent iff \( Z_n \) is positive recurrent iff it has an invariant distribution \( \pi(n) \) so that
\[ \pi(n) = \sum \pi(m)p(m, n) = \frac{1}{2}(\pi(n-1) + \pi(n+2)) \]
This is a linear recurrence with fundamental solution
\[ \pi(n) = c^n \]
where
\[ c = \frac{1}{2}(1 + c^3) \]
This gives
\[ c^3 - 2c + 1 = 0 \]
This factors as
\[ (c - 1)(c^2 + c - 1) = 0 \]
So,
\[ c = 1, -1 \pm \frac{\sqrt{5}}{2} =, 1, 0.618, -1.618 \]
The second number gives a solution since it gives \(0 < c < 1\). Then
\[ \pi(n) = \frac{c^n}{1 - c} \]
is an invariant distribution.

The equation for the invariant distribution in terms of \(A\) is:
\[ \sum_m \pi(m)\alpha(m, n) = 0 \]
\[ 2\pi(n - 1) + 2\pi(n + 2) - 4\pi(n) = 0 \]
which also gives
\[ c^3 - 2c + 1 = 0. \]

Both of these problems were positive recurrent. You should also be prepared for transient, null-recurrent and explosive chains.