MATH 56A: STOCHASTIC PROCESSES
PRACTICE QUIZ 2 ANSWERS

Problem 1 Consider the countable Markov chain with state space

$$S = \{0, 1, 2, 3, \cdots\}$$

and probabilities $p(n, n+1) = p, p(n, 0) = q = 1 - p$.

For which values of p is this chain positive recurrent, null recurrent, transient?

The first point you need to understand is: This chain is transient if $p = 1$ because, in that case, you keep increasing with probability one. The chain is not even irreducible.

If $p < 1$ then there is always a positive probability q of going to 0. So, you keep coming to 0 with probability one. So, $p < 1$ implies recurrent.

X_n is *positive recurrent* if it has an invariant distribution π. This is a probability vector satisfying:

$$\pi(n) = \sum \pi(m)p(m, n)$$

If $n > 0$ there is only one term in the sum:

$$\pi(n) = \pi(n-1)p = \pi(n-2)p^2 = \cdots = \pi(0)p^n$$

If $n = 0$ then

$$\pi(0) = \sum_{n=0}^{\infty} \pi(n)q = (1)q = q$$

So,

$$\pi(n) = qp^n$$

The sum of these numbers is

$$\sum \pi(n) = \sum_{n=0}^{\infty} qp^n = \frac{q}{1-p} = \frac{1}{1} = 1$$

So, everything checks out and we see that X_n is positive recurrent when $p < 1$. It is transient if $p = 1$ and it is never null recurrent.

Problem 2 Consider the continuous Markov chain X_t with states

$$S = \{0, 1, 2, 3, \cdots\}$$

and transition rates given by $\alpha(n, n+1) = 2, \alpha(n, n-2) = 2$.

(a) Convert this to a countable chain Z_n.

This is very easy. Since the rates are equal we have:

$$p(n, n+1) = 1/2, \quad p(n, n-2) = 1/2.$$

(b) Show that X_t is xxx positive recurrent. (Oops, it *is* positive recurrent.)

X_t is positive recurrent iff Z_n is positive recurrent iff it has an invariant distribution $\pi(n)$ so that

$$\pi(n) = \sum \pi(m)p(m, n) = \tfrac{1}{2}(\pi(n-1) + \pi(n+2))$$

This is a linear recurrence with fundamental solution

$$\pi(n) = c^n$$

where

$$c = \tfrac{1}{2}(1 + c^3)$$

This gives

$$c^3 - 2c + 1 = 0$$

This factors as

$$(c - 1)(c^2 + c - 1) = 0$$

So,

$$c = 1, \frac{-1 \pm \sqrt{5}}{2} =, 1, 0.618, -1.618$$

The second number gives a solution since it gives $0 < c < 1$. Then

$$\pi(n) = \frac{c^n}{1 - c}$$

is an invariant distribution.

The equation for the invariant distribution in terms of A is:

$$\sum_m \pi(m)\alpha(m, n) = 0$$

$$2\pi(n - 1) + 2\pi(n + 2) - 4\pi(n) = 0$$

which also gives

$$c^3 - 2c + 1 = 0.$$

Both of these problems were positive recurrent. You should also be prepared for transient, null-recurrent and explosive chains.