

MATH 56A: STOCHASTIC PROCESSES
QUIZ 2

Problem 1 [“Double or nothing”] Suppose that $S = \{0, 1, 2, 3, 4, \dots\}$ with transition probabilities:

$$p(n, 2n) = \frac{2}{3}, \quad p(n, 0) = \frac{1}{3}, \quad p(0, 1) = 1$$

and all other $p(n, m) = 0$.

a) If $X_0 \neq 0$ what is the probability that X_1, X_2, \dots, X_n are all nonzero? Conclude that, with probability one, you will eventually reach 0.

$$\mathbb{P}(X_1, \dots, X_n \neq 0 \mid X_0 \neq 0) = \prod_{i=1}^n \mathbb{P}(X_i \neq 0 \mid X_{i-1} \neq 0) = \left(\frac{2}{3}\right)^n$$

Since this number goes to zero as $n \rightarrow \infty$, the probability that you never go to zero is zero.

b) Find the communication classes. (Recall that x, y are in the same class if you can get from x to y and from y to x .)

This has a strange answer: The numbers

$$0, 1, 2, 4, 8, 16, \dots$$

(0 and all powers of 2) form one communication class. Call it C . Every other number is in its own communication class!

c) Which communications classes are transient, which are null recurrent and which are positive recurrent?

All of the singleton communication classes are transient. The infinite class C is positive recurrent.

d) Find the invariant distribution for each positive recurrent communication class.

To find the invariant distribution, you have to solve the equation:

$$\pi(n) = \sum \pi(m)p(m, n)$$

For $n = 2^k, k \geq 2$ this is

$$\pi(2^k) = \pi(2^{k-1})\frac{2}{3} = \pi(1) \left(\frac{2}{3}\right)^k$$

and

$$\pi(1) = \pi(0).$$

Since the sum of the $\pi(n)$ must be 1 you get:

$$1 = \pi(1) \left(1 + 1 + \frac{2}{3} + \left(\frac{2}{3}\right) + \dots\right) = \pi(1) \left(1 + \frac{1}{1 - 2/3}\right) = 4\pi(1)$$

So,

$$\pi(0) = \pi(1) = \frac{1}{4}$$

and

$$\pi(2^k) = \frac{2^{k-2}}{3^k}$$

e) If $X_0 = 0$ find the expected return time to 0.

The expected return time to 0 is

$$\frac{1}{\pi(0)} = 4.$$

Problem 2 [Hint: look on page 78 of your book.] Consider the continuous birth-death chain with birth and death rates:

$$\lambda_n = \frac{1}{n+1}, \quad \mu_n = 1$$

a) Show that this chain is positive recurrent.

The theorem (on p.78) is that this chain is positive recurrent if the series

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

But this is equal to

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

So, the chain is positive recurrent.

b) Find the invariant distribution $\pi(n)$.

Again, the book says:

$$\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \pi(0) = \frac{1}{n!} \pi(0)$$

but

$$1 = \sum_{n=0}^{\infty} \pi(n) = \pi(0) \sum_{n=0}^{\infty} \frac{1}{n!} = \pi(0)e$$

So, $\pi(0) = 1/e$ and

$$\pi(n) = \frac{1}{n!} e^{-1}$$

c) Find $A = (\alpha(n, m))$ (What is the formula for $\alpha(n, n)$?) and verify that your invariant distribution π is a left null eigenvector of the infinitesimal generator A . (i.e., check to see if your answer is correct!)

The infinitesimal generator is given by $A = (\alpha(n, m))$ where

$$\alpha(n, n+1) = \frac{1}{n+1}, \quad \alpha(n, n-1) = 1, \quad \alpha(n, n) = -\frac{n+2}{n+1}$$

for $n \geq 1$ and

$$\alpha(0, 1) = 1, \quad \alpha(0, 0) = -1$$

with all other numbers being zero. We want to verify that $\pi A = 0$. In other words,

$$\sum_m \pi(m) \alpha(m, n) \stackrel{?}{=} 0$$

$$\pi(n-1) \alpha(n-1, n) + \pi(n+1) \alpha(n+1, n) + \pi(n) \alpha(n, n) \stackrel{?}{=} 0$$

$$\pi(n-1) \frac{1}{n} + \pi(n+1) \stackrel{?}{=} \pi(n) \left(\frac{n+2}{n+1} \right)$$

Now plug in $\pi(n) = \frac{1}{n!}e^{-1}$:

$$\frac{e^{-1}}{n!} + \frac{e^{-1}}{(n+1)!} \stackrel{=?}{=} \frac{e^{-1}}{n!} \left(\frac{n+2}{n+1} \right)$$

Multiply by e to get

$$\begin{aligned} \frac{1}{n!} + \frac{1}{(n+1)!} &\stackrel{=?}{=} \frac{1}{n!} \left(\frac{n+2}{n+1} \right) \\ \frac{n+1}{(n+1)!} + \frac{1}{(n+1)!} &= \frac{n+2}{(n+1)!} \stackrel{=?}{=} \frac{1}{n!} \left(\frac{n+2}{n+1} \right) \end{aligned}$$

So, it is true. (The answer to part (b) is really an invariant probability distribution.)