

**MATH 56A: STOCHASTIC PROCESSES
PRACTICE QUIZ**

Problem 1 Consider the countable Markov chain with state space

$$S = \{0, 1, 2, 3, \dots\}$$

and probabilities $p(n, n+1) = n/(n+1)$, $p(n, n-1) = 1/(n+1)$.

a) **Determine if this chain is transient or recurrent.** This Markov Chain is transient. We can see this heuristically (verbally) or computationally.

Verbal argument The reason is by comparison with the known transient chain given by

$$p(n, n+1) = 2/3, \quad p(n, n-1) = 1/3$$

(with $p(0, 1) = 1$). Since $p = 2/3 > 1/2$ we know that this is transient. So, there is a positive probability that we will never reach 0 (we go to ∞). But,

$$\frac{n}{n+1} \geq \frac{2}{3}$$

for $n \geq 2$. So, the given chain has a greater probability of going to the right than the known transient chain. So, it also has a positive probability of never reaching 0, once it gets past 1. Therefore, this chain is transient.

Computational proof: To see if this is transient, we should look for a function $a(n)$ so that $a(0) = 1$ and

$$a(n) = \sum p(n, m)a(m) \text{ for } n > 0$$

In this case this is:

$$a(n) = p(n, n+1)a(n+1) + p(n, n-1)a(n-1)$$

$$a(n) = \frac{n}{n+1}a(n+1) + \frac{1}{n+1}a(n-1)$$

$$(n+1)a(n) = na(n+1) + a(n-1)$$

$$a(n) - a(n-1) = n(a(n+1) - a(n))$$

$$a(n+1) - a(n) = \frac{1}{n}(a(n) - a(n-1)) = \frac{1}{n} \frac{1}{n-1} \frac{1}{n-2} \cdots \frac{1}{1}(a(1) - a(0)) = \frac{a(1) - a(0)}{n!}$$

$$a(n+1) = a(n) + \frac{1}{n!}(a(1) - a(0)) = a(n-1) + \frac{1}{(n-1)!}(a(1) - a(0)) + \frac{1}{n!}(a(1) - a(0))$$

$$= a(0) + \left(1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)(a(1) - a(0))$$

$$\lim_{n \rightarrow \infty} a(n+1) = 1 + e(a(1) - 1)$$

We want this to be zero. So

$$e(1 - a(1)) = 1$$

$$a(1) = 1 - 1/e$$

b) **What is the statement of the theorem you are using?** (You might want to prepare this kind of answer in advance.)

The computational argument uses the theorem:

Theorem 0.0.1. *An irreducible countable Markov chain is transient if there is a function $a(n)$ with*

- (1) $a(0) = 1$ (or $a(z) = 1$ for some other point z)
- (2) $0 \leq a(n) \leq 1$ for all n
- (3) $a(n) = \sum_m p(n, m)a(m)$
- (4) There is some n so that $a(n) < 1$ (or, equivalently, $\inf a(n) = 0$).

The verbal argument is some kind of comparison argument.

c) **If recurrent determine if it is positive recurrent or null recurrent. What is the formula you are using?** Doesn't apply. If the numbers were reversed: $p(n, n+1) = 1/(n+1)$, $p(n, n-1) = n/(n+1)$ then we could use the same kind of argument (comparison with the case $p(n, n+1) = 1/3$, $p(n, n-1) = 2/3$ to show that it is positive recurrent.

Problem 2 Consider the birth-death problem with

$$\lambda_n = np^n = \text{birth rate when population is } n$$

and no deaths.

For which values of p will there be an explosion?

The expected time it takes to go from n to $n+1$ is $1/\lambda_n = \frac{1}{np^n}$. So, we have an explosion (with probability one) if

$$\sum_{n=1}^{\infty} \frac{1}{np^n} < \infty$$

This series converges if and only if $p > 1$. This is when the population will explode.