MATH 56A: STOCHASTIC PROCESSES WORKSHEET

2. Answers to Worksheet 2

$$P = \begin{pmatrix} .8 & 0 & .2 & 0 & 0 \\ 0 & .5 & 0 & 0 & .5 \\ .3 & 0 & .7 & 0 & 0 \\ .2 & 0 & 0 & .6 & .2 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(1) Calculate the rank of $P - I_5$ by *column reduction.*

$$P - I = \begin{pmatrix} -.2 & 0 & .2 & 0 & 0 \\ 0 & -.5 & 0 & 0 & .5 \\ .3 & 0 & -.3 & 0 & 0 \\ .2 & 0 & 0 & -.4 & .2 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

The first step is to clear the last column. This is a column operation given by adding the first 4 columns to the last column:

$$\begin{pmatrix} -.2 & 0 & .2 & 0 & 0 \\ 0 & -.5 & 0 & 0 & 0 \\ .3 & 0 & -.3 & 0 & 0 \\ .2 & 0 & 0 & -.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Divide the fourth column by $-.4$ and use it to clear the fourth row:

$$\begin{pmatrix} -.2 & 0 & .2 & 0 & 0 \\ 0 & -.5 & 0 & 0 & 0 \\ .3 & 0 & -.3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Now add the first column to the 3rd column:

$$\begin{pmatrix} -.2 & 0 & 0 & 0 & 0 \\ 0 & -.5 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

This is good enough because each row has only one nonzero entry.

(2) How many recurrent classes does this Markov chain have?

The rank of $P - I$ is equal to 3 (the number of nonzero columns in the reduced form). Therefore, the nullity is $5 - 3 = 2$. (This is the number of zero columns in the reduced form.) So, there are exactly 2 recurrent classes.

(3) Find the left null vectors of $P - I_5$. Normalize to get the *basic invariant distributions*.

The left null vectors are:

$$(3, 0, 2, 0, 0), \quad (0, 2, 0, 0, 1).$$

These are the solutions of the equations

$$(-.2)x_1 + .3x_3 = 0$$

$$(-.5)x_2 + (1)x_5 = 0$$

$$x_4 = 0$$

which come from the matrix equation:

$$(x_1, x_2, x_3, x_4, x_5)\begin{pmatrix} -.2 & 0 & 0 & 0 & 0 \\ 0 & -.5 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = (0, 0, 0, 0, 0).$$

When you normalized these two vectors, you get the basic invariant probability distributions:

$$\pi_1 = (\tfrac{3}{5}, 0, \tfrac{2}{5}, 0, 0), \quad \pi_2 = (0, \tfrac{2}{3}, 0, 0, \tfrac{1}{3})$$

(4) What are the *supports* of these vectors? These are the recurrent classes.

The supports are the coordinates which are nonzero. These are:

$$\text{supp}(\pi_1) = \{1, 3\} = R_1, \quad \text{supp}(\pi_2) = \{2, 5\} = R_2.$$

(5) Find all invariant distributions π.

These are given by the formula:

$$\pi = t\pi_1 + (1-t)\pi_2 = \left(\frac{3t}{5}, \frac{2(1-t)}{3}, \frac{2t}{5}, 0, \frac{1-t}{3}\right)$$

where $0 \le t \le 1$.

(6) If the initial distribution is

$$\alpha = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right),$$

what is the invariant distribution

$$\alpha P^\infty := \lim_{n\to\infty} \alpha P^n \quad ?$$

To answer this we need to determine t, the probability of ending up in recurrent state R_1. Since 2 and 3 are in R_2, R_1 respectively, we start in each of those recurrent classes with probability 1/3. For the remaining 1/3rd of the time we start in the transient state 4. We need to look at the 4th row of the original matrix P to see what happens in that case:

$$(.2, 0, 0, .6, .2).$$

This show a loop at 4. When you escape that loop you go with equal probability to 1 and 5 which are in the recurrent classes R_1, R_2. So, you end up in R_1, R_2 later with probability $1/6, 1/6$ resp. In total, the probability is $1/2, 1/2$ that you end up in R_1, R_2. So,

$$t = \mathbb{P}(X_n \in R_1 \text{ for large } n) = \frac{1}{2}.$$

So,

$$\alpha P^\infty = \pi = \left(\frac{3}{10}, \frac{1}{3}, \frac{1}{5}, 0, \frac{1}{6}\right).$$

(7) Renumber the states and put the matrix P into *canonical form.*

The recurrent states should come first. To avoid confusion, I will use letters to indicate the reordering:

$$a = 1, b = 3, c = 2, d = 5, e = 4.$$

Then the matrix P becomes:

$$\begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{pmatrix} = \left(\begin{array}{cc|cc|c} .8 & .2 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 \\ \hline 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline .2 & 0 & 0 & .2 & .6 \end{array}\right)$$