MATH 56A: STOCHASTIC PROCESSES
WORKSHEET

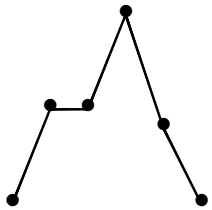
3. Worksheet 3

The purpose of this worksheet is to practice the iteration algorithm, plot a convex value function and determine the optimal stopping time.

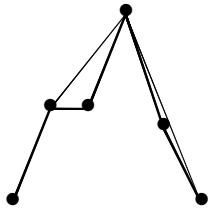
We have a simple random walk with states $0, 1, 2, 3, 4, 5$ where the endpoints are absorbing. The payoff function is:

$$f = (0, 5, 5, 10, 4, 0)$$

(1) Plot the function $f(x)$ and connects the dots.



(2) Find the value function $v(x)$. This is the convex hull of the graph of $f(x)$.



$$v = (0, 5, 7.5, 10, 5, 0)$$

(3) Find the superharmonic

$$u_1(x) = \begin{cases} 0 & \text{if } x \text{ is absorbing} \\ max\ f(y) & \text{if } x \text{ is transient} \end{cases}$$

$$u_1 = (0, 10, 10, 10, 10, 0)$$

(4) Find $u_2, u_3, \cdots$ The general formula is, for x transient,

$$u_{n+1}(x) = \max(f(x), \sum p(x, y)u_n(y))$$

For random walk this becomes:

$$u_{n+1}(x) = \max\left(f(x), \frac{u_n(x-1) + u_n(x+1)}{2}\right)$$

If I call the second term $ave(u_n)$ then:

$$ave(u_1) = (0, 5, 10, 10, 5, 0)$$

$$u_2 = \max(f, ave(u_1)) = (0, 5, 10, 10, 5, 0)$$

$$ave(u_2) = (0, 5, 7.5, 7.5, 5, 0)$$

$$u_3 = \max(f, ave(u_2)) = (0, 5, 7.5, 10, 5, 0)$$
$$ave(u_3) = (0, 3.25, 7.5, 6.25, 5, 0)$$
$$u_4 = \max(f, ave(u_3)) = (0, 5, 7.5, 10, 5, 0)$$

(5) Show that u_n converges to v as $n \to \infty$.

What I intended to ask in this problem is: Show that u_n *converges*.

Since $u_4 = u_3$, the formula will give us $u_n = u_3$ for all $n \geq 3$. So, the sequence converges to u_3. The theorem is that the limit is $v(x) = \lim u_n(x)$.

The proof of this, in this case, is as follows.

First or all,

$$v(x) \leq u_3(x).$$

The intuitive reason for this is that each u_n is an optimistic estimate for the value function and therefore greater or equal to the true value function v.

Since $v(x) \geq f(x)$, the value function satisfies

$$v(x) = \max(f(x), \frac{v(x-1) + v(x+1)}{2}) \geq \max(f(x), \frac{f(x-1) + f(x+1)}{2}) = u_3(x)$$

So,

$$v(x) \geq u_3(x) \geq v(x) \quad \Rightarrow \quad v(x) = u_3(x).$$