

## MATH 56A: STOCHASTIC PROCESSES HOMEWORK

From the syllabus: There will be weekly homework. Students are encouraged to work on their homework in groups and to access all forms of aid including expert advice, internet and other resources. The work you hand in should, however, be in your own words and in your own handwriting. Every student hands in his own homework. There will be a penalty for late homework.

### HOMEWORK 1B FINITE MARKOV CHAINS

This homework project is due next Thursday, Feb 14:

Your assignment is to analyze the Leontief economic model and work out one example. Don't do the calculations by hand.

In the Leontief model, there are "factories" which require the output of other factories to make their product. For each dollar of output, Factory  $i$  requires  $q_{ij}$  dollars worth of output of factory  $j$ . The total amount that factory  $i$  needs to spend for each dollar of output is

$$q_{i1} + q_{i2} + \cdots + q_{ir} \leq 1.$$

We always assume the sum is  $\leq 1$ . (But it is allowed to be equal to 1.)

Work out the following example and answer the questions (in complete sentences so that your kid brother can read it!)

**Example:** We have 4 factories:

$S$  = Steel

$W$  = Water

$E$  = Coal/Gas/Oil

$P$  = Plastic

To produce \$1 worth of steel, the steel factory needs 50¢ worth of energy and 25¢ worth of water (and no plastic). This goes into the matrix  $Q$  in the first line.

$$Q = \begin{pmatrix} 0 & .25 & .5 & 0 \\ .1 & 0 & .4 & .2 \\ .2 & .1 & .3 & .1 \\ .1 & .2 & .2 & .1 \end{pmatrix}$$

Each factory keeps a stockpile of material, say 10\$ worth of each item. When it get an order for goods, the factory uses its inventory and then orders replacements. So, the steel factory, after filling out an order for 1\$ worth of steel will order 25¢ worth of water and 50¢ worth of energy.

(\*) Write in words: What do the numbers in the fourth row of the matrix mean?

The consumer wants 1\$ of steel, 2\$ of water, 10\$ of energy and 2\$ of plastic.

(a) How much does each factory need to make?

(b) Follow the money: Where do the 15\$ go after 4 rounds?

(c) How long does it take for all factories to regain 99% of their original inventory assuming that they keep 10\$ worth of each commodity in stock.

(d) Follow the energy. Take the total amount of energy (your answer to part (a)) that is needed. Where does it go?

**Theory:** Assume we have a more general matrix  $Q$  representing the requirements of each industry in the Leontief model.

(e) The rows of  $Q$  may not add up to less than 1. If row  $i$  adds up to 1, what does it mean about state  $i$ ? Give an example.

(f) Suppose that every row of  $Q$  adds up to at most  $p = 0.9$ . Then prove that each row of  $Q^n$  adds up to

$$p^n = (0.9)^n$$

or less.

Why does this imply that the sequence

$$I + Q + Q^2 + Q^3 + \dots$$

converges? [Hint: a series (infinite sum) of matrices converges if and only if, for each  $i$  and  $j$  the sum of the  $(i, j)$  entries converges. Use the comparison test, comparing these entries to a geometric series to show that the series converges.]