MATH 56A: STOCHASTIC PROCESSES
HOMEWORK

Homework 5
Martingales

These problems are due Wednesday, March 26. Answers will be posted the following week.

First problem: Suppose that $X_1, X_2, \cdots$ are i.i.d. random variables with expected value $\mu = \mathbb{E}(X_i)$.

a) Show that

$$M_n = \frac{X_1 X_2 \cdots X_n}{\mu^n}$$

is a martingale wrt $X_1, X_2, \cdots$

b) Assuming that $X_i > 0$, is $\ln M_n$ (natural log of M_n) a martingale in general? [See next question.]

c) Modify $\ln M_n$ to make it into a martingale.

Second problem: (a special case of the first problem)

d) Suppose that (in the above problem)

$$X_i = \begin{cases} 1/2 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2 \end{cases}$$

Then show that $T =$ first time that $M_n = 1/2$ is a stopping time. Does the Optimal Sampling Theorem hold?

e) Using the Central Limit Theorem (on $\ln M_n$), show that, with probability one, M_n converges to 0. Explain why this contradicts the Martingale convergence theorem. (What part of the statement holds? What part doesn't hold?)

Third problem Suppose that

$$X_i = \begin{cases} -1 & \text{with probability } 2/3 \\ 2 & \text{with probability } 1/3 \end{cases}$$

f) Show that

$$S_n = X_1 + X_2 + \cdots + X_n$$

is a martingale wrt $X_1, X_2, \cdots$?

g) Show that the Optimal Sampling Theorem does not hold. [Let $T =$ first time that $S_n > 0$. Show that S_n is a recurrent Markov Chain and therefore $\mathbb{P}(T < \infty) = 1$]

h) Is $S_n/\sqrt{n}$ a martingale wrt $X_1, X_2, \cdots$?