MATH 56A: STOCHASTIC PROCESSES HOMEWORK

Homework 8
Brownian motion

Three problems are due on the last day of class. Answers will be posted the following week.

First problem: (reflection principle) Let W_t be standard Brownian motion. Calculate the probability that there exist $0 < a < b < c < 1$ so that $W_a = 1, W_b = -1, W_c = 0$. Give the details of the argument. Use scaling to find the probability that there exist $0 < a < b < c < t$ so that $W_a = 1, W_b = -1, W_c = 0$.

Second problem: (fractal dimension) Take the unit interval $[0, 1]$ and remove the open sets $(1/4, 3/8)$ and $(5/8, 3/4)$. In other word, you remove two open intervals $1/8$ of a unit long leaving three closed intervals $1/4$ of a unit long. Repeat the process infinitely often. Each time you remove two open pieces from each interval that you have leaving three closed intervals which are exactly $1/4$ the size of the interval. Find the fractal dimension of the resulting set using the two methods taught in class:

(1) By scaling. (This is the method we used to compute the fractal dimension of the Cantor set which is very similar to this set.)
(2) By cutting up the interval into smaller intervals and counting how many intervals we need to cover the set. (This is the definition of the box dimension.)

Note: Your two answers should be equal.

Third problem: (heat equation) Suppose that B is the infinite horizontal strip:

$$B = \{(x, y) \in \mathbb{R}^2 \mid |y| < 1\}$$

Let g be the function on the boundary of B given by

$$g(x, \pm 1) = x$$

(1) Solve the heat equation $\Delta f = 0$. [This is very easy. Just guess. Use the fact that the solution is unique.]
(2) Write the solution as a probability statement. $\mathbb{E}^x(...) = ...$
(3) Give a probabilistic argument to prove this. [This uses a reflection type argument on an unknown density function.]