

**MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK**

0. HOMEWORK 0 ANSWERS

0.1. In the Kermack-McKendrick model, prove that the number of infected reaches its highest point when the number of susceptibles is equal to the threshold (or at $t = 0$ at the beginning of the recorded/modeled time period).

This is easy. You just need to look at the notes to see what the equations are.

The number of infected is

$$y = N - x_0 e^{-z/\rho} - z$$

where N is the total population, x_0 is the initial number of infected, z is the number of removed and ρ is the threshold, a positive constant.

Since $d^2y/dz^2 < 0$, the infected curve is concave down and reaches its highest point either at one endpoint or at the point where $dy/dz = 0$. The epidemic cannot stop when $y > 0$ because the infected are continually being removed. The numbers cannot stabilize until $y = 0$. So, the maximum is either at the beginning (at $t = 0, z = 0$) or when $dy/dz = 0$. But

$$\frac{dy}{dz} = \frac{x_0}{\rho} e^{-z/\rho} - 1 = \frac{x}{\rho} - 1$$

since the number of susceptible is

$$x = x_0 e^{-z/\rho}.$$

So, $dy/dz = 0$ if and only if $x = \rho$.

If the maximum occurs at the beginning then the epidemic is in its final stages and the maximum actually occurred earlier before we were recording the data.

0.2. Find all functions $x(t), y(t)$ so that

$$x'(t) = 5x - y, y'(t) = 3x + y$$

Find the particular solution with initial position $(x_0, y_0) = (1, 3)$.

The matrix is

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

This has eigenvalues 4, 2 with corresponding eigenvectors $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. So $A = QDQ^{-1}$ where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad Q^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

And

$$e^{tA} = Qe^{tD}Q^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{4t} - e^{2t} & -e^{4t} + e^{2t} \\ 3e^{4t} - 3e^{2t} & -e^{4t} + 3e^{2t} \end{pmatrix}$$

The general solution is $X = e^{tA}X_0$ or

$$x = \frac{x_0}{2} (3e^{4t} - e^{2t}) + \frac{y_0}{2} (-e^{4t} + e^{2t})$$

$$y = \frac{x_0}{2} (3e^{4t} - 3e^{2t}) + \frac{y_0}{2} (-e^{4t} + 3e^{2t})$$

When $x_0 = 1, y_0 = 3$ then the e^{4t} terms all cancel and we get

$$x = e^{2t}, \quad y = 3e^{2t}.$$

0.3. Find all functions f from integers to complex numbers so that

$$f(n+1) = 4f(n) - 5f(n-1).$$

Now find the solution when $f(0) = f(1) = 2$ and explain why it is real.

To solve the homogeneous equation, try $f = c^n$. If c is a double root then the second solution is $f(n) = nc^n$. The equation is

$$c^2 - 4c + 5 = 0$$

This has two complex roots:

$$c = 2 \pm i$$

So, the two linearly independent complex solutions are

$$f_+(n) = (2+i)^n$$

$$f_-(n) = (2-i)^n$$

The general complex solution is therefore given by

$$f(n) = a(2+i)^n + b(2-i)^n$$

where $a, b \in \mathbb{C}$.

Given the initial conditions $f(0) = f(1) = 2$ we get:

$$f(0) = a + b = 2$$

or: $b = 2 - a$.

$$f(1) = a(2+i) + b(2-i) = a(2+i) + (2-a)(2-i) = 2$$

The solution is:

$$a = 1 + i, \quad b = 1 - i$$

So,

$$f(n) = (1+i)(2+i)^n + (1-i)(2-i)^n.$$

This is a sequence of real number, in fact integers, since they are Gaussian integers which are equal to their own conjugates. (*Gaussian integers* are complex numbers of the form $a + ib$ where $a, b \in \mathbb{Z}$.)

0.4. Find the function $f(n), n = 0, 1, 2, 3, \dots$ so that $f(0) = 0$

$$f(n) = \frac{1}{3} [f(n-1) + f(n+1) + f(n+2)], \quad n \geq 1$$

$$\lim_{n \rightarrow \infty} f(n) = 1.$$

The general solutions are $f(n) = c^n$ where c is a solution of

$$c^3 + c^2 - 3c + 1 = 0$$

After factoring:

$$c^3 + c^2 - 3c + 1 = (c-1)(c^2 + 2c - 1).$$

So, $c = 1, -1 \pm \sqrt{2}$. The general solution is:

$$f(n) = A + B(-1 + \sqrt{2})^n + C(-1 - \sqrt{2})^n$$

The condition $f(0) = 0$ implies that

$$A + B + C = 0.$$

Since $\lim_{n \rightarrow \infty} f(n) = 1$ it also follows that

$$\lim_{n \rightarrow \infty} f(2n) = 1$$

but,

$$\lim_{n \rightarrow \infty} (-1 - \sqrt{2})^{2n} = \infty$$

since $|-1 - \sqrt{2}| > 1$ and

$$\lim_{n \rightarrow \infty} (-1 + \sqrt{2})^{2n} = 0$$

since $|-1 + \sqrt{2}| < 1$. Therefore, in order to have $\lim_{n \rightarrow \infty} f(n) = 1$, we must have $C = 0$ and $A = 1$. So, $B = -1$ and

$$f(n) = 1 - (-1 + \sqrt{2})^n.$$