MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK

Answers to Homework 1a

0.1. Do 1.5 in the book.

There are two recurrent classes: $R_1 = \{0, 1\}$ and $R_2 = \{2, 4\}$ and one transient class $T = \{3, 5\}$.

Since 0 is in the recurrent class R_1, if you start at 0 you will keep going back to it over and over. The invariant distribution for R_1 is

$$\pi_1 = (3/8, 5/8)$$

So,

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) = \frac{3}{8}.$$

If you start at $X_0 = 5$ then the answer, using what I taught you so far, is given by taking the $(5, 0)$ coordinate of P^∞ which you can calculate using a computer.

To do it by hand, you need to use the next topic in the course, which I will do Monday. The formula is given by:

$$(I - Q)^{-1} S_1 = \begin{pmatrix} 7/11 \\ 6/11 \end{pmatrix} = \begin{pmatrix} \mathbb{P}(X_\infty \in R_1 \,|\, X_0 = 3) \\ \mathbb{P}(X_\infty \in R_1 \,|\, X_0 = 5) \end{pmatrix}$$

Here

$$Q = \begin{pmatrix} 0 & .25 \\ .2 & .4 \end{pmatrix}, \quad I - Q = \begin{pmatrix} 1 & -.25 \\ -.2 & .6 \end{pmatrix}$$

$$(I - Q)^{-1} = \frac{1}{55}\begin{pmatrix} 60 & 25 \\ 20 & 100 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 12 & 5 \\ 4 & 20 \end{pmatrix}, \quad S_1 = \begin{pmatrix} .5 \\ .2 \end{pmatrix}$$

S_1, S_2 are given by combining the states of each recurrent class into one state, giving a new transition matrix:

$$\widetilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ S_1 & S_2 & Q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .5 & .25 & 0 & .25 \\ .2 & .2 & .2 & .4 \end{pmatrix}$$

So, the final answer is:

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 5) = \frac{6}{11}\frac{3}{8} = \frac{9}{44}.$$

0.2.

A man is playing two slot machines (call them A and B). Machine A gives a payoff with a probability of 1/6 and machine B gives a payoff with probability 1/16. The man starts by picking a machine at random. Then he plays the machine until he has lost twice (not in a row, just in total). Then he switches machines and continues. For example, suppose that his winning (1) and losing (0) sequence might be:

$${}_0 1_1 0_2 0_3 1_4 0_5 1_6 0_7 1_8 0_9 1_{10} 0_{11} 0_{12}$$

Then he will switch machines after $n = 2$ since he lost twice. (He switches in the time between $n = 2$ and $n = 3$). He switches back after $n = 6$ and then again after $n = 10$.

(a) Make this into a Markov chain with 4 states: A_0, A_1, B_0, B_1 where the subscript keeps track of the number of losses. [This is an example of recording information to convert a stochastic process to first order.]

The states are:

A_0: The man is playing machine A and has not lost any games so far.

A_1: The man is playing machine A and has lost one gave so far.

B_0: The man is playing machine B and has not lost any games so far.

B_1: The man is playing machine B and has lost one gave so far.

The transition matrix is:

$$P = \begin{pmatrix} 1/6 & 5/6 & 0 & 0 \\ 0 & 1/6 & 5/6 & 0 \\ 0 & 0 & 1/16 & 15/16 \\ 15/16 & 0 & 0 & 1/16 \end{pmatrix}.$$

"The man starts by picking a machine at random" means the initial distribution is

$$\alpha = (1/2, 0, 1/2, 0)$$

since he starts at one of the two machines with equal probability but he starts with no losses.

(b) What is the probability that the man will be playing machine A at $n = 4$ if he starts at machine A? What about if he starts at a machine picked at random?

$$P^2 = \begin{pmatrix} \frac{1}{36} & \frac{10}{36} & \frac{25}{36} & 0 \\ 0 & \frac{1}{36} & \frac{55}{288} & \frac{5\cdot 15}{6\cdot 16} \\ \frac{15^2}{16^2} & 0 & \frac{1}{16^2} & \frac{30}{16^2} \\ \frac{15}{6\cdot 16} + \frac{15}{16^2} & \frac{5\cdot 15}{6\cdot 16} & 0 & \frac{1}{16^2} \end{pmatrix}.$$

The sum of the first two columns (indicating the probability of playing on machine A after two rounds of play) is

$$(11/36, 1/36, 15^2/16^2, 1 - 1/16^2)^t.$$

The answer to the first question is given by the dot product of this with the first row (which indicates the probability distribution after two rounds of play):

$$\frac{11}{36^2} + \frac{10}{36^2} + \frac{25 \cdot 15^2}{36 \cdot 16^2} \approx 0.62655527$$

The dot product with the third row gives the probability of playing on machine A after 4 rounds of play starting at machine B:

$$\approx 0.38871765$$

The answer to the second question is the average of these two numbers:

$$\tfrac{1}{2}(0.62655527 + 0.38871765) \approx 0.50763646$$

(c) Find the invariant distribution.

The invariant distribution is proportional to

$$(6/5, 6/5, 16/15, 16/15) = \tfrac{2}{15}(9, 9, 8, 8).$$

So, it is

$$\pi = \tfrac{1}{34}(9, 9, 8, 8).$$

(d) In the long run, how much of the time will the man be playing the better machine?

The answer is the sum of the first two coordinates of the invariant distribution:

$$\frac{9}{34} + \frac{9}{34} = \frac{9}{17}.$$

0.3. Suppose that

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ .4 & .4 & 0 & .2 \\ .7 & 0 & 0 & .3 \end{pmatrix}$$

(a) Find the unique invariant distribution and explain why it is unique.

The unique recurrent class is $R = \{1, 4\}$. So, the unique invariant distribution is

$$\pi = (7/17, 0, 0, 10/17).$$

(b) Draw the diagram and find the communication classes.

Draw arrows: $1 \to 4 \to 1, 2 \to 3 \to 2$ and $3 \to 1, 3 \to 4$.

The communication classes are the recurrent class $R = \{1, 4\}$ and the transient class $T = \{2, 3\}$.

(c) What is the probability that X_{100} is in the transient class given that you start in the transient class?

You need to go back and forth between states 2 and 3 fifty times. So,

$$\mathbb{P}(X_{100} \in T \,|\, X_0 \in T) = 0.4^{50} \approx 1.26765 * 10^{-20}.$$

What about if you start at a random location?

In that case the probability is half of this:

$$\mathbb{P}(X_{100} \in T) = \tfrac{1}{2}0.4^{50} \approx 6.33825 * 10^{-21}.$$