

MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK

HOMEWORK 1B ANSWERS

This homework project is due Thursday, Feb 14:

Your assignment is to analyze the Leontief economic model and work out one example. Don't do the calculations by hand.

In the Leontief model, there are "factories" which require the output of other factories to make their product. For each dollar of output, Factory i requires q_{ij} dollars worth of output of factory j . The total amount that factory i needs to spend for each dollar of output is

$$q_{i1} + q_{i2} + \cdots + q_{ir} \leq 1.$$

We always assume the sum is ≤ 1 . (But it is allowed to be equal to 1.)

Work out the following example and answer the questions (in complete sentences so that your kid brother can read it!)

Example: We have 4 factories:

S = Steel

W = Water

E = Coal/Gas/Oil

P = Plastic

To produce \$1 worth of steel, the steel factory needs 50¢ worth of energy and 25¢ worth of water (and no plastic). This goes into the matrix Q in the first line.

$$Q = \begin{pmatrix} 0 & .25 & .5 & 0 \\ .1 & 0 & .4 & .2 \\ .2 & .1 & .3 & .1 \\ .1 & .2 & .2 & .1 \end{pmatrix}$$

Each factory keeps a stockpile of material, say 10\$ worth of each item. When it get an order for goods, the factory uses its inventory and then orders replacements. So, the steel factory, after filling out an order for 1\$ worth of steel will order 25¢ worth of water and 50¢ worth of energy.

(*) **Write in words: What do the numbers in the fourth row of the matrix mean?**

For every dollar's worth of plastic that it produces, the plastic factory needs 10¢ worth of steel, 20¢ worth of water, 20¢ worth of energy and 10¢ worth of plastic.

The consumer wants 1\$ of steel, 2\$ of water, 10\$ of energy and 2\$ of plastic.

(a) **How much does each factory need to make?**

The production vector is

$$(1, 2, 10, 2)(I + Q + Q^2 + Q^3 + \cdots) = (1, 2, 10, 2)(I - Q)^{-1}$$

$$= \left(\frac{307800}{39675}, \frac{315750}{39675}, \frac{1045500}{39675}, \frac{274500}{39675} \right)$$

$$= \left(\frac{4104}{529}, \frac{4210}{529}, \frac{13940}{529}, \frac{3660}{529} \right)$$

The coordinates tell how much steel, water, energy and plastic need to be produced.

(b) **Follow the money: Where do the 15\$ go after 4 rounds?**

After 4 round, the location of the money is given by

$$(1, 2, 10, 2)Q^4 = (.651525, 6306, 1.70725, .4925)$$

for a total of \$3.481875 held by the factories and the remaining \$11.518125 in the bank. The term “4 rounds” could also be interpreted as $(1, 2, 10, 2)Q^3$.

(c) **How long does it take for all factories to regain 99% of their original inventory assuming that they keep 10\$ worth of each commodity in stock.**

Since 99% of \$10 is \$9.90, this question is asking what is the smallest integer n so that all four coordinates of $(1, 2, 10, 2)Q^n$ are ≤ 0.1 (10¢)

Since

$$(1, 2, 10, 2)Q^{11} = (0.05, 0.05, 0.13, 0.04)$$

$$(1, 2, 10, 2)Q^{12} = (0.03, 0.03, 0.09, 0.03),$$

the answer is 12 days (or whatever the time unit is).

(d) **Follow the energy. Take the total amount of energy (your answer to part (a)) that is needed. Where does it go?**

The amount of energy used by the steel factory is

$$\frac{4104}{529}(.5) = \frac{2052}{529}$$

For water, energy and plastic it is:

$$\frac{4210}{529}(.4) = \frac{1684}{529}$$

$$\frac{13940}{529}(.3) = \frac{4182}{529}$$

$$\frac{3660}{529}(.2) = \frac{732}{529}$$

The total amount of energy that is used in this process is, in dollars,

$$\frac{13940}{529}$$

the remaining \$10 worth of energy is used by the consumer.

Theory: Assume we have a more general matrix Q representing the requirements of each industry in the Leontief model.

(e) The rows of Q may not add up to less than 1. If row i adds up to 1, what does it mean about state i ? Give an example.

This means that factory i does not make a profit. Here is a really simple example:

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Here factory A gives all of its income to B and is officially nonprofit. Factory B (very suspiciously) has no expenses and puts all its income in the bank.

Here is another example.

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here we have two nonprofit factories. To produce one dollar worth of stuff, each factory needs one dollar worth of the other factory's stuff. If you buy something from them, it creates an infinite amount of work for them since the money goes back and forth and never goes to the bank! So, mathematically speaking, nonprofit organizations are extremely wasteful and inefficient!

(f) Suppose that every row of Q adds up to at most $p = 0.9$. Then prove that each row of Q^n adds up to

$$p^n = (0.9)^n$$

or less.

The proof is by induction. The statement is true for $n = 1$ by assumption. Suppose it is true for n . This means that each row of $X = Q^n$ adds up to at most p^n . In other words,

$$\sum_j x_{ij} \leq p^n$$

Then the i th row of $Q^{n+1} = XQ$ adds up to

$$\sum_{j,k} x_{ij}q_{jk} = \sum_j x_{ij} \sum_k q_{jk} \leq \sum_j x_{ij}p \leq p^n p = p^{n+1}.$$

Therefore, the statement holds for $n + 1$. So, it holds for all $n \geq 1$ by induction. We used the fact that the entries x_{ij} are nonnegative. (So $a \leq b \Rightarrow x_{ij}a \leq x_{ij}b$.)

Why does this imply that the sequence

$$I + Q + Q^2 + Q^3 + \dots$$

converges? [Hint: a series (infinite sum) of matrices converges if and only if, for each i and j the sum of the (i, j) entries converges. Use the comparison test, comparing these entries to a geometric series to show that the series converges.]

Since the geometric series

$$1 + p + p^2 + \dots$$

converges to

$$\frac{1}{1-p} = 1/.1 = 10$$

and each entry of Q^n is nonnegative and $\leq p^n$, the sum of the (i, j) entries of the matrices I, Q, Q^2, \dots converges by the comparison test to a nonnegative number $\leq 1/(1-p) = 10$.