

**MATH 56A: STOCHASTIC PROCESSES  
ANSWERS TO HOMEWORK**

HOMEWORK 2 ANSWERS

1) The first problem is the queueing problem.

a) [When is this positive recurrent?](#)

We need to find the invariant distribution  $\pi$ . This is the solution of

$$\pi(n) = \sum_m \pi(m)p(m, n)$$

and

$$\sum \pi(n) = 1.$$

For  $n > 0$ , the first equation gives:

$$\pi(n) = p(1 - q)\pi(n - 1) + (qp + (1 - p)(1 - q))\pi(n) + q(1 - p)\pi(n + 1)$$

Here it helps to use some notation:

$$a = p(1 - q), \quad b = q(1 - p), \quad c = qp + (1 - p)(1 - q)$$

These numbers add up to 1 since they are the numbers in the  $n$ th row of the transition matrix.

$$a + b + c = 1.$$

So,

$$\begin{aligned} \pi(n) &= a\pi(n - 1) + c\pi(n) + b\pi(n + 1) \\ (1 - c)\pi(n) &= (a + b)\pi(n) = a\pi(n - 1) + b\pi(n + 1) \\ \pi(n) &= \frac{a}{a + b}\pi(n - 1) + \frac{b}{a + b}\pi(n + 1) \end{aligned}$$

To solve this you put

$$\pi(n) = r^n$$

Then

$$r = \frac{a}{a + b} + \frac{b}{a + b}r^2$$

So, either  $r = 1$  or

$$r = a/b = \frac{p(1 - q)}{q(1 - p)}$$

An invariant distribution exists if and only if  $r < 1$  which is equivalent to  $p < q$ . In that case, the [invariant distribution](#) is:

$$\pi(n) = C \left(\frac{a}{b}\right)^n \quad \text{for } n > 0$$

and

$$\pi(0) = C(1 - q)$$

This last equation comes from:

$$\begin{aligned}\pi(0) &= \pi(0)p(0,0) + \pi(1)p(1,0) = (1-p)\pi(0) + b\pi(1) \\ p\pi(0) &= b\pi(1) = bC(a/b) = aC = p(1-q)C. \\ \pi(0) &= (1-q)C.\end{aligned}$$

To find  $C$ , we take the sum:

$$\begin{aligned}\sum \pi(n) &= C(1-q + (a/b) + (a/b)^2 + \dots) = C\left(\frac{1}{1-a/b} - q\right) \\ 1 &= C\left(\frac{b}{b-a} - \frac{q(q-p)}{q-p}\right) = C\frac{q-q^2}{q-p} \\ C &= \frac{q-p}{q-q^2}\end{aligned}$$

This makes

$$\pi(0) = (1-q)C = \frac{q-p}{q} = 1 - p/q.$$

And, for  $n > 0$  we have:

$$\pi(n) = C(a/b)^n = \frac{(q-p)p^n(1-q)^n}{(q-q^2)q^n(1-p)^n} = \frac{(q-p)p^n(1-q)^{n-1}}{q^{n+1}(1-p)^n}$$

b) **When is this chain transient?**

Now we have to find a function  $\alpha(n)$  so that  $\alpha(0) = 0$ ,

$$\alpha(n) = \sum p(n,m)\alpha(m)$$

and

$$\inf \alpha(n) = 0.$$

This gives:

$$\begin{aligned}\alpha(n) &= p(n,n-1)\alpha(n-1) + p(n,n)\alpha(n) + p(n,n+1)\alpha(n+1) \\ \alpha(n) &= b\alpha(n-1) + c\alpha(n) + a\alpha(n+1) \\ (1-c)\alpha(n) &= (a+b)\alpha(n) = b\alpha(n-1) + a\alpha(n+1)\end{aligned}$$

This is the same equation as for  $\pi(n)$  except that  $a, b$  are switched. Thus, the chain is transient iff  $b < a$  iff  $q < p$ . The value of the function  $\alpha(n)$  for all  $n$  is

$$\alpha(n) = C(b/a)^n$$

The constant  $C$  must be equal to 1 to make this agree with  $\alpha(0) = 1$ . So,

$$\alpha(n) = \frac{q^n(1-p)^n}{p^n(1-p)^n}$$

c) In the remaining case,  $p = q$ , the chain must be null-recurrent.

2) The second problem is easy.

a) **When is this chain positive recurrent?**

As I explained in class, the chain is positive recurrent if and only if the expected return time (to any state) is finite. Take the state 0. Then, you go to state  $x$  with probability  $p_x$  and then return to 0 in  $x$  more steps. So, the expected value of the return time  $T$  is

$$\mathbb{E}(T) = \sum_{n=1}^{\infty} (n+1)p_n.$$

Since  $\sum p_n = 1$  this is

$$\mathbb{E}(T) = 1 + \sum_{n=1}^{\infty} np_n.$$

So, the chain is positive recurrent if and only if

$$\sum_{n=1}^{\infty} np_n < \infty.$$

b) Find the invariant distribution.

Let

$$L = \sum_{n=1}^{\infty} np_n < \infty.$$

Then,

$$\pi(0) = \frac{1}{\mathbb{E}(T)} = \frac{1}{L+1}.$$

What about  $\pi(n)$  for  $n > 0$ ?

The equation is

$$\begin{aligned} \pi(y) &= \sum \pi(x)p(x,y) \\ \pi(y) &= \pi(0)p(0,y) + \pi(y+1)p(y+1,y) = \pi(0)p_y + \pi(y+1) \\ \Delta_n &= \pi(n+1) - \pi(n) = -\pi(0)p_n = -\frac{p_n}{1+L} \\ \Delta_{n-1} &= \pi(n) - \pi(n-1) = -\pi(0)p_{n-1} = -\frac{p_{n-1}}{1+L} \end{aligned}$$

Adding these up (and using  $p_0 = 0$ ) we get:

$$\begin{aligned} \pi(n+1) - \pi(0) &= -\frac{p_1 + p_2 + \dots + p_n}{1+L} \\ \pi(n+1) &= \pi(0) - \frac{p_1 + p_2 + \dots + p_n}{1+L} = \frac{1 - p_1 - p_2 - \dots - p_n}{1+L} \\ \pi(n) &= \frac{1 - p_1 - p_2 - \dots - p_{n-1}}{1+L} \end{aligned}$$

Most students divided by the sum instead of using the value of  $\pi(0)$ . This gives a messy but correct answer.

2.14. a) Explain why  $\phi'(a) < 1$ .

This is obvious from the graph. The rigorous proof is by the mean value theorem. Since  $\phi(a) = a$  and  $\phi(1) = 1$ , there is a number  $b$  between  $a$  and 1 so that  $\phi'(b) = 1$ . Since  $\phi''(x) > 0$  for all  $x$  this implies that  $\phi'(x) < 1$  for all  $x < b$  in particular  $\phi'(a) < 1$ .

b) Show that, for  $n$  sufficiently large

$$a - a_{n+1} \leq \rho(a - a_n)$$

for some  $\rho < 1$ .

The answer is:  $\rho = \phi'(a) < 1$ . Since  $\phi(x)$  is concave up and  $\phi(a_n) = a_{n+1}$ , the point  $(a_n, a_{n+1})$  lies on the graph of  $\phi$  which lies above the tangent line to  $\phi$  at the point  $(a, a)$  which is given by the equation:

$$y - a = \rho(x - a)$$

Putting in  $x = a_n$  we get that the point  $(a_n, a + \rho(a_n - a))$  is on the tangent line. So,

$$a_{n+1} \geq a + \rho(a_n - a)$$

Subtract  $a$  from both sides to get:

$$a_{n+1} - a \geq \rho(a_n - a)$$

Change sign to get:

$$a - a_{n+1} \leq \rho(a - a_n)$$

c) Show that, for some  $b > 0, c < \infty$ ,

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) \leq ce^{-bn}$$

The answer is that  $b = -\ln \rho$ , i.e.,

$$\rho = e^{-b}$$

Since  $\rho < 1, b > 0$ . Also,

$$c = \frac{1}{1 - a}$$

Here is the proof: The conditional probability is:

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) = \frac{\mathbb{P}(\text{extinction and } X_n \neq 0)}{\mathbb{P}(X_n \neq 0)} = \frac{a - a_n}{1 - a_n}.$$

First the denominator:

$$\begin{aligned} a_n &< a \\ 1 - a_n &> 1 - a \\ \frac{1}{1 - a_n} &< \frac{1}{1 - a} = c \end{aligned}$$

Next, the numerator:

$$a - a_n \leq \rho^n$$

This is by induction on  $n$ . For  $n = 0$  we have:

$$a - a_0 = a < 1 = \rho^0.$$

Suppose that the statement is true for  $n$ . Then

$$a - a_{n+1} \leq \rho(a - a_n) \leq \rho\rho^n = \rho^{n+1}$$

So, the statement holds for all  $n$ . So,

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) = \frac{a - a_n}{1 - a_n} \leq \frac{\rho^n}{1 - a} = c\rho^n = ce^{-bn}.$$

2.16. a) Show that, for  $n > 0$ ,

$$\pi(n) = \sum_{m=n-1} \pi(m)p_{n-m}$$

The definition of invariant distribution gives:

$$\pi(n) = \sum \pi(m)p(m, n) = \sum \pi(m)p_{n-m}$$

the only question is: Over what value of  $m$  is this sum taken? But that is easy since  $p_x$  is only defined for  $x \leq 1$ . So we must have

$$n - m \leq 1$$

$$n - 1 \leq m.$$

b) Let  $q_k = p_{1-k}$ . Then show that there is some  $\alpha \in (0, 1)$  so that

$$\alpha = \phi(\alpha) = q_0 + q_1\alpha + q_2\alpha^2 + \dots$$

The number  $\alpha$  is the extinction probability. It will be less than 1 if the average number of children is greater than 1. Call this number  $\mu^*$  since there is already a  $\mu$  in the problem given by

$$\mu = \sum np_n < 0.$$

The average number of offspring is

$$\mu^* = \sum kq_k = \sum kp_{1-k}$$

But  $\sum p_{1-k} = 1$ . So,

$$1 - \mu^* = \sum (p_{1-k} - kp_{1-k}) = \sum (1 - k)p_{1-k} = \mu.$$

So,

$$\mu^* = 1 - \mu > 1.$$

c) Use the  $\alpha$  from part (b) to find the invariant distribution.

Start with

$$\alpha = \sum_{k=0}^{\infty} q_k \alpha^k = \sum_{k=0}^{\infty} p_{1-k} \alpha^k = \sum_{m=n-1}^{\infty} p_{n-m} \alpha^{1-n+m}$$

Now multiply both sides by  $\alpha^{n-1}$  to get:

$$\alpha^n = \sum_{m=n-1}^{\infty} p_{n-m} \alpha^m$$

This implies that

$$\pi(n) = C\alpha^n$$

for  $n > 0$ . To find  $C$  we add this up:

$$C = \frac{1}{\sum \alpha^n} = 1 - \alpha$$

So,

$$\pi(n) = (1 - \alpha)\alpha^n$$

This is a probability distribution. After one generation we get a new probability distribution

$$\pi P = (?, \pi(1), \pi(2), \dots)$$

The first coordinate must be  $\pi(0)$  since the numbers add up to 1. So, this is the invariant probability distribution.