1) The first problem is the queueing problem.

a) When is this positive recurrent?

We need to find the invariant distribution \( \pi \). This is the solution of

\[
\pi(n) = \sum_{m} \pi(m)p(m,n)
\]

and

\[
\sum \pi(n) = 1.
\]

For \( n > 0 \), the first equation gives:

\[
\pi(n) = p(1-q)\pi(n-1) + (qp + (1-p)(1-q))\pi(n) + q(1-p)\pi(n+1)
\]

Here it helps to use some notation:

\[
a = p(1-q), \quad b = q(1-p), \quad c = qp + (1-p)(1-q)
\]

These numbers add up to 1 since they are the numbers in the \( n \)th row of the transition matrix.

\[
a + b + c = 1.
\]

So,

\[
\pi(n) = a\pi(n-1) + c\pi(n) + b\pi(n+1)
\]

\[
(1-c)\pi(n) = (a + b)\pi(n) = a\pi(n-1) + b\pi(n+1)
\]

\[
\pi(n) = \frac{a}{a+b}\pi(n-1) + \frac{b}{a+b}\pi(n+1)
\]

To solve this you put

\[
\pi(n) = r^n
\]

Then

\[
r = \frac{a}{a+b} + \frac{b}{a+b}r^2
\]

So, either \( r = 1 \) or

\[
r = \frac{a}{b} = \frac{p(1-q)}{q(1-p)}
\]

An invariant distribution exists if and only if \( r < 1 \) which is equivalent to \( p < q \). In that case, the invariant distribution is:

\[
\pi(n) = C \left( \frac{a}{b} \right)^n \quad \text{for} \quad n > 0
\]

and

\[
\pi(0) = C(1-q)
\]
This last equation comes from:

$$\pi(0) = \pi(0)p(0,0) + \pi(1)p(1,0) = (1-p)\pi(0) + b\pi(1)$$

$$p\pi(0) = b\pi(1) = bC(a/b) = aC = p(1-q)C.$$  

$$\pi(0) = (1-q)C.$$

To find $C$, we take the sum:

$$\sum \pi(n) = C \left(1 - q + \frac{a}{b} + \frac{a}{b}^2 + \cdots\right) = C \left(\frac{1}{1-a/b} - q\right)$$

$$1 = C \left(\frac{b}{b-a} - \frac{q(q-p)}{q-p}\right) = C \frac{q-q^2}{q-p}$$

$$C = \frac{q-p}{q-q^2}$$

This makes

$$\pi(0) = (1-q)C = \frac{q-p}{q} = 1 - p/q.$$  

And, for $n > 0$ we have:

$$\pi(n) = C(a/b)^n = \frac{(q-p)p^n(1-q)^n}{(q-q^2)q^n(1-p)^n} = \frac{(q-p)p^n(1-q)^{n-1}}{q^{n+1}(1-p)^n}$$

b) When is this chain transient?  
Now we have to find a function $\alpha(n)$ so that $\alpha(0) = 0$,

$$\alpha(n) = \sum p(n,m)\alpha(m)$$

and

$$\inf \alpha(n) = 0.$$  

This gives:

$$\alpha(n) = p(n,n-1)\alpha(n-1) + p(n,n)\alpha(n) + p(n,n+1)\alpha(n+1)$$

$$\alpha(n) = ba\alpha(n-1) + ca\alpha(n) + a\alpha(n+1)$$

$$(1-c)\alpha(n) = (a+b)\alpha(n) = ba\alpha(n-1) + aa\alpha(n+1)$$

This is the same equation as for $\pi(n)$ except that $a, b$ are switched. Thus, the chain is transient iff $b < a$ iff $q < p$. The value of the function $\alpha(n)$ for all $n$ is

$$\alpha(n) = C(b/a)^n$$

The constant $C$ must be equal to 1 to make this agree with $\alpha(0) = 1$. So,

$$\alpha(n) = \frac{q^n(1-p)^n}{p^n(1-p)^n}$$

c) In the remaining case, $p = q$, the chain must be null-recurrent.

2) The second problem is easy.  

a) When is this chain positive recurrent?
As I explained in class, the chain is positive recurrent if and only if the expected return time (to any state) is finite. Take the state 0. Then, you go to state \( x \) with probability \( p_x \) and then return to 0 in \( x \) more steps. So, the expected value of the return time \( T \) is

\[
E(T) = \sum_{n=1}^{\infty} (n + 1)p_n.
\]

Since \( \sum p_n = 1 \) this is

\[
E(T) = 1 + \sum_{n=1}^{\infty} np_n.
\]

So, the chain is positive recurrent if and only if

\[
\sum_{n=1}^{\infty} np_n < \infty.
\]

b) Find the invariant distribution.

Let

\[
L = \sum_{n=1}^{\infty} np_n < \infty.
\]

Then,

\[
\pi(0) = \frac{1}{E(T)} = \frac{1}{L+1}.
\]

What about \( \pi(n) \) for \( n > 0 \)?

The equation is

\[
\pi(y) = \sum \pi(x)p(x, y) = \pi(0)p(0, y) + \pi(y + 1)p(y + 1, y) = \pi(0)p_y + \pi(y + 1)
\]

\[
\Delta_n = \pi(n + 1) - \pi(n) = -\pi(0)p_n = -\frac{p_n}{1+L}
\]

\[
\Delta_{n-1} = \pi(n) - \pi(n - 1) = -\pi(0)p_{n-1} = -\frac{p_{n-1}}{1+L}
\]

Adding these up (and using \( p_0 = 0 \)) we get:

\[
\pi(n + 1) - \pi(0) = -\frac{p_1 + p_2 + \ldots + p_n}{1+L}
\]

\[
\pi(n + 1) = \pi(0) - \frac{p_1 + p_2 + \ldots + p_n}{1+L} = \frac{1 - p_1 - p_2 - \ldots - p_n}{1+L}
\]

\[
\pi(n) = \frac{1 - p_1 - p_2 - \ldots - p_{n-1}}{1+L}
\]

Most students divided by the sum instead of using the value of \( \pi(0) \). This gives a messy but correct answer.

2.14. a) Explain why \( \phi'(a) < 1 \).

This is obvious from the graph. The rigorous proof is by the mean value theorem. Since \( \phi(a) = a \) and \( \phi(1) = 1 \), there is a number \( b \) between \( a \) and 1 so that \( \phi'(b) = 1 \). Since \( \phi''(x) > 0 \) for all \( x \) this implies that \( \phi'(x) < 1 \) for all \( x < b \) in particular \( f'(a) < 1 \).

b) Show that, for \( n \) sufficiently large

\[
a - a_{n+1} \leq \rho(a - a_n)
\]
for some $\rho < 1$.

The answer is: $\rho = \phi'(a) < 1$. Since $\phi(x)$ is concave up and $\phi(a_n) = a_{n+1}$, the point $(a_n, a_{n+1})$ lies on the graph of $\phi$ which lies above the tangent line to $\phi$ at the point $(a, a)$ which is given by the equation:

$$y - a = \rho(x - a)$$

Putting in $x = a_n$ we get that the point $(a_n, a + \rho(a_n - a))$ is on the tangent line. So,

$$a_{n+1} \geq a + \rho(a_n - a)$$

Subtract $a$ from both sides to get:

$$a_{n+1} - a \geq \rho(a_n - a)$$

Change sign to get:

$$a - a_{n+1} \leq \rho(a - a_n)$$

c) Show that, for some $b > 0, c < \infty$,

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) \leq ce^{-bn}$$

The answer is that $b = -\ln \rho$, i.e.,

$$\rho = e^{-b}$$

Since $\rho < 1, b > 0$. Also,

$$c = \frac{1}{1 - a}$$

Here is the proof: The conditional probability is:

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) = \frac{\mathbb{P}(\text{extinction and } X_n \neq 0)}{\mathbb{P}(X_n \neq 0)} = \frac{a - a_n}{1 - a_n}.$$  

First the denominator:

$$a_n < a$$

$$1 - a_n > 1 - a$$

$$\frac{1}{1 - a_n} < \frac{1}{1 - a} = c$$

Next, the numerator:

$$a - a_n \leq \rho^n$$

This is by induction on $n$. For $n = 0$ we have:

$$a - a_0 = a < 1 = \rho^0.$$  

Suppose that the statement is true for $n$. Then

$$a - a_{n+1} \leq \rho(a - a_n) \leq \rho \rho^n = \rho^{n+1}$$

So, the statement holds for all $n$. So,

$$\mathbb{P}(\text{extinction} \mid X_n \neq 0) = \frac{a - a_n}{1 - a_n} \leq \frac{\rho^n}{1 - a} = c \rho^n = ce^{-bn}.$$  

2.16. a) Show that, for $n > 0$,

$$\pi(n) = \sum_{m=n-1}^{\infty} \pi(m)p_{n-m}$$
The definition of invariant distribution gives:
\[ \pi(n) = \sum \pi(m)p(m,n) = \sum \pi(m)p_{n-m} \]
the only question is: Over what value of \( m \) is this sum taken? But that is easy since \( p_x \) is only defined for \( x \leq 1 \). So we must have
\[ n - m \leq 1 \]
\[ n - 1 \leq m. \]

b) Let \( q_k = p_{1-k} \). Then show that there is some \( \alpha \in (0,1) \) so that
\[ \alpha = \phi(\alpha) = q_0 + q_1\alpha + q_2\alpha^2 + \cdots \]

The number \( \alpha \) is the extinction probability. It will be less than 1 if the average number of children is greater than 1. Call this number \( \mu^* \) since there is already a \( \mu \) in the problem given by
\[ \mu = \sum np_n < 0. \]
The average number of offspring is
\[ \mu^* = \sum kq_k = \sum kp_{1-k} \]
But \( \sum p_{1-k} = 1 \). So,
\[ 1 - \mu^* = \sum (p_{1-k} - kp_{1-k}) = \sum (1-k)p_{1-k} = \mu. \]
So,
\[ \mu^* = 1 - \mu > 1. \]

c) Use the \( \alpha \) from part (b) to find the invariant distribution.
Start with
\[ \alpha = \sum_{k=0}^{\infty} q_k\alpha^k = \sum_{k=0}^{\infty} p_{1-k}\alpha^k = \sum_{m=n-1}^{\infty} p_{n-m}\alpha^{1-n+m} \]
Now multiply both sides by \( \alpha^{n-1} \) to get:
\[ \alpha^n = \sum_{m=n-1}^{\infty} p_{n-m}\alpha^m \]
This implies that
\[ \pi(n) = C\alpha^n \]
for \( n > 0 \). To find \( C \) we add this up:
\[ C = \frac{1}{\sum \alpha^n} = 1 - \alpha \]
So,
\[ \pi(n) = (1 - \alpha)\alpha^n \]
This is a probability distribution. After one generation we get a new probability distribution
\[ \pi P = (?, \pi(1), \pi(2), \cdots) \]
The first coordinate must be \( \pi(0) \) since the numbers add up to 1. So, this is the invariant probability distribution.