MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK

Homework 3
Continuous Markov Chains

Three problems due 6pm Monday, March 10. Answers will be posted Tuesday evening.

Quiz 2 on Friday, March 14.

3.4 A is the infinitesimal generator for an irreducible continuous time Markov chain with finite state space.

a) Let a be a positive number greater (in absolute value) than all the entries of A. Let

$$P = (1/a)A + I$$

Show that P is the transition matrix for a discrete time, irreducible aperiodic Markov chain.

Since the rows of A add up to 0, the rows of P add up to 1:

$$\sum_j p(i,j) = \sum_j (1/a)\alpha(i,j) + \delta(i,j) = (1/a)\sum_j \alpha(i,j) + \sum_j \delta(i,j) = (1/a)(0) + 1.$$

The entries of P are nonnegative:

$$p(i,j) = (1/a)\alpha(i,j) \geq 1 \text{ if } i \neq j$$

$$p(i,i) = (1/a)\alpha(i,i) + 1 \geq 1 - (1/a)|\alpha(i,i)| > 1 - 1 = 0 \text{ since } a > |\alpha(i,i)|.$$

Since the diagonal entries of P are positive, the chain is aperiodic.

Since $p(i,j) > 0$ whenever $\alpha(i,j) > 0$, the communication classes for P are the same as for A. So, the discrete Markov chain is irreducible given that the continuous one is irreducible.

b) Show that A has a unique left eigenvector with eigenvalue 0 that is a probability vector and all the other eigenvalues of A have real part strictly less than 0.

This follows from the Perron-Frobenius Theorem (p.17):

First of all the eigenvalues of P, A are related as follows: If λ is an eigenvalue of P with left eigenvector x (and right eigenvector y) then

$$xP = \lambda x$$

So,

$$xA = x[aP - aI] = axP - ax = a\lambda x - ax = (a\lambda - a)x$$

which makes $a(\lambda - 1)$ an eigenvalue of A. [You can also use the right eigenvector y to get the same conclusion.]

When $\lambda = 1$, the corresponding eigenvector for A is $a(\lambda - 1) = 0$.

"1 is a simple eigenvalue of P" means there is a unique left eigenvector with eigenvalue 1:

$$\pi P = \pi$$

This is the invariant probability distribution. The calculation above implies that π is also the unique left eigenvector of A with eigenvalue 0. The eigenvector π is a "probability vector" since its entries are nonnegative and add up to 1.

The other eigenvalues of A are $a(\lambda - 1)$ where $|\lambda| < 1$. But this implies that the real part of λ has absolute value less than 1. Since a is positive real this implies that

$$\Re(a(\lambda - 1)) = a(\Re\lambda - 1) < 0.$$

3.9

The infinitesmal generator is

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(a) Find the invariant distribution π.

This is the solution of $\pi A = 0$ normalized so that the sum of the coordinates is 1:

$$\pi = \frac{1}{8}(1, 3, 2, 2)$$

(b) If $X_0 = 1$ what is the expected amount of time until the first jump?

The change rate is 2 times per unit time. So, the expected wait is 1/2 of a unit of time.

(c) If $X_0 = 1$ what is the expected time until you reach state 4?

You take $\tilde{A} = A$ with the 4th row and 4th column deleted

$$\tilde{A} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -3 \end{pmatrix}$$

Then $b(x) =$ (expected time to get from x to 4) is given by the formula (Example 3 on page 74):

$$b = [-\tilde{A}]^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1/2 & 5/2 & 1 \\ 1/2 & 3/2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

So the answer is

$$b(1) = 4.$$

(Using the corrected version of the explosion probability calculation we did in class) Let $b(x)$ be the expected time that it takes to get from state x to state 4. Then,

$$b(x) = \frac{1}{|a(x,x)|} + \sum_{y \neq x, 4} p(x,y) b(y)$$

The first number is the time it takes to escape from state x. If you jump to a state y which is not 4 then you need more time and the amount of extra time you need is $b(y)$ with probability $p(x,y)$. For the matrix A we get:

$$b(1) = \frac{1}{2} + \frac{1}{2}(b(2) + b(3))$$

$$b(2) = 1 + b(3)$$
$$b(3) = \frac{1}{3} + \frac{1}{3}(b(1) + b(2))$$

The solution of this system of equations is

$$b = (4, 4, 3)$$

as before.

($M/M/1$ queue).

Suppose that there is a queue with one server. People get into the line at a rate of λ and they get served at the rate of μ.

(1) This is a continous Markov chain X_t with states $0, 1, 2, 3, \cdots$. What is the infinitesimal generator $A = (\alpha(x, y))$?

This is given by

$$a(n, n+1) = \lambda \text{ for } n \geq 0$$
$$a(n, n-1) = \mu \text{ for } n > 0$$

and don't forget:

$$a(n, n) = -\lambda - \mu \text{ for } n \geq 0$$

The other entries of A are all zero.

(2) Convert this to a countable Markov chain Z_n. What is the (infinite) probability transition matrix $P = (p(x, y))$?

$$p(n, n+1) = \frac{\lambda}{\lambda + \mu} \text{ for } n > 0$$
$$p(n, n-1) = \frac{\mu}{\lambda + \mu} \text{ for } n > 0$$
$$p(0, 1) = 1$$

and the other entries of P are zero. These are the probabilities for the jumps between states.

$$p(x, y) = \mathbb{P}(\text{the state jumps to } y \text{ from } x)$$

(3) Using your answers to Homework Problem #2.1 determine

(a) Under what conditions is this queue transient, positive recurrent, null recurrent?

This is a little tricky. The conversion is

$$p = \frac{\lambda}{1 + \lambda} = 1 - \frac{1}{1 + \lambda}$$
$$q = \frac{\mu}{1 + \mu} = 1 - \frac{1}{1 + \mu}$$

Since the function $f(x) = 1 - 1/(1 + x)$ has positive derivative:

$$f'(x) = \frac{1}{(1 + x)^2} > 0$$

So, $f(x)$ is monotonically increasing. I.e., $p > q \iff \lambda > \mu$ and $p < q \iff \lambda < \mu$. So this countable Markov chain is

(i) transient iff $p > q$ iff $\lambda > \mu$
(ii) null recurrent iff $p = q$ iff $\lambda = \mu$
(iii) positive recurrent iff $p < q$ iff $\lambda < \mu$

To prove this you have to also convert 2.1 into jump probabilities.

$$p(x,x+1) = \mathbb{P}(\text{the state jumps to } x+1 \text{ from } x) = \frac{p(1-q)}{p(1-q)+q(1-p)} = \frac{\lambda}{\lambda+\mu}$$

$$p(x,x-1) = \mathbb{P}(\text{the state jumps to } x-1 \text{ from } x) = \frac{q(1-p)}{p(1-q)+q(1-p)} = \frac{\mu}{\lambda+\mu}$$

(b) When it is positive recurrent, what is the expected return time to 0? (For X_t not Z_n).

The invariant distribution (for Z_n, not for X_n) is given by normalizing the sequence:

$$\left(\frac{\lambda}{\lambda+\mu}, \frac{\lambda}{\mu}, \frac{\lambda^2}{\mu^2}, \cdots\right)$$

The sum of these numbers is

$$\frac{\lambda}{\lambda+\mu} + \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{2\lambda\mu}{\mu^2-\lambda^2}$$

So,

$$\pi(0) = \frac{\mu^2-\lambda^2}{2\lambda\mu}\frac{\lambda}{\lambda+\mu} = \frac{\mu-\lambda}{2\mu}$$

So, the expected return time to 0 for Z_n is

$$\frac{1}{\pi(0)} = \frac{2\mu}{\mu-\lambda} = 1 + \frac{\lambda+\mu}{\mu-\lambda}$$

(Note that this number is ≥ 2 since it takes at least 2 steps to go from 0 back to 0.) For the continuous chain X_t, the first step takes $1/\lambda$ amount of time and every subsequent step takes $1/(\lambda+\mu)$ amount of time. So the correct answer is:

$$\mathbb{E}(T) = \frac{1}{\lambda} + \left(\frac{\lambda+\mu}{\mu-\lambda}\right)\left(\frac{1}{\mu+\lambda}\right) = \frac{1}{\lambda} + \frac{1}{\mu-\lambda}$$

(I wonder if anyone got this.)

(4) Determine when the chain X_t is explosive.

This chain is never explosive for the simple reason that each jump takes an average of

$$\frac{1}{\lambda+\mu}$$

amount of time. So, you can't have an infinite number of jumps in a finite amount of time. In particular, you can't go to infinity. Another method is to show that the amount of time it takes to get from state n to state $n+1$ is bounded below by $\frac{1}{\lambda+\mu}$. So, the infinite sum diverges and you have no explosion.