Homework 3  
Continuous Markov Chains

Three problems due 6pm Monday, March 10. Answers will be posted Tuesday evening. 
Quiz 2 on Friday, March 14.

3.4 A is the infinitesimal generator for an irreducible continuous time Markov chain with finite state space.

a) Let $a$ be a positive number greater (in absolute value) than all the entries of $A$. Let 

$$ P = \frac{1}{a}A + I $$

Show that $P$ is the transition matrix for a discrete time, irreducible aperiodic Markov chain.

Since the rows of $A$ add up to 0, the rows of $P$ add up to 1:

$$ \sum_j p(i,j) = \sum_j (\frac{1}{a}\alpha(i,j) + \delta(i,j)) = (\frac{1}{a})\sum_j \alpha(i,j) + \sum_j \delta(i,j) = (\frac{1}{a})(0) + 1. $$

The entries of $P$ are nonnegative:

$$ p(i,j) = (\frac{1}{a})\alpha(i,j) \geq 1 \text{ if } i \neq j $$

$$ p(i,i) = (\frac{1}{a})\alpha(i,i) + 1 \geq 1 - (\frac{1}{a})|\alpha(i,i)| > 1 - 1 = 0 \text{ since } a > |\alpha(i,i)|. $$

Since the diagonal entries of $P$ are positive, the chain is aperiodic.

Since $p(i,j) > 0$ whenever $\alpha(i,j) > 0$, the communication classes for $P$ are the same as for $A$. So, the discrete Markov chain is irreducible given that the continuous one is irreducible.

b) Show that $A$ has a unique left eigenvector with eigenvalue 0 that is a probability vector and all the other eigenvalues of $A$ have real part strictly less than 0.

This follows from the Perron-Frobenius Theorem (p.17):

First of all the eigenvalues of $P, A$ are related as follows: If $\lambda$ is an eigenvalue of $P$ with left eigenvector $x$ (and right eigenvector $y$) then

$$ xP = \lambda x $$

So,

$$ xA = x[aP - aI] = axP - ax = a\lambda x - ax = (a\lambda - a)x $$

which makes $a(\lambda - 1)$ an eigenvalue of $A$. [You can also use the right eigenvector $y$ to get the same conclusion.]

When $\lambda = 1$, the corresponding eigenvector for $A$ is $a(\lambda - 1) = 0$.

"1 is a simple eigenvalue of $P"$ means there is a unique left eigenvector with eigenvalue 1:

$$ \pi P = \pi $$

1
This is the invariant probability distribution. The calculation above implies that \(\pi\) is also the unique left eigenvector of \(A\) with eigenvalue 0. The eigenvector \(\pi\) is a “probability vector” since its entries are nonnegative and add up to 1.

The other eigenvalues of \(A\) are \(a(\lambda - 1)\) where \(|\lambda| < 1\). But this implies that the real part of \(\lambda\) has absolute value less than 1. Since \(a\) is positive real this implies that \(\Re(a(\lambda - 1)) = a(\Re\lambda - 1) < 0\).

3.9

The infinitesimal generator is

\[
A = \begin{pmatrix}
-2 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 1 & -3 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

(a) Find the invariant distribution \(\pi\).

This is the solution of \(\pi A = 0\) normalized so that the sum of the coordinates is 1:

\[
\pi = \frac{1}{8}(1, 3, 2, 2)
\]

(b) If \(X_0 = 1\) what is the expected amount of time until the first jump?

The change rate is 2 times per unit time. So, the expected wait is 1/2 of a unit of time.

(c) If \(X_0 = 1\) what is the expected time until you reach state 4?

You take \(\tilde{A} = A\) with the 4th row and 4th column deleted

\[
\tilde{A} = \begin{pmatrix}
-2 & 1 & 1 \\
0 & -1 & 1 \\
1 & 1 & -3
\end{pmatrix}
\]

Then \(b(x) = \) (expected time to get from \(x\) to 4) is given by the formula (Example 3 on page 74):

\[
b = [-\tilde{A}]^{-1} \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1/2 & 5/2 & 1 \\ 1/2 & 3/2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4/3 \\ 4/3 \end{pmatrix}
\]

So the answer is

\[
b(1) = 4.
\]

(Using the corrected version of the explosion probability calculation we did in class) Let \(b(x)\) be the expected time that it takes to get from state \(x\) to state 4. Then,

\[
b(x) = \frac{1}{|a(x, x)|} + \sum_{y \neq x, 4} p(x, y)b(y)
\]

The first number is the time it takes to escape from state \(x\). If you jump to a state \(y\) which is not 4 then you need more time and the amount of extra time you need is \(b(y)\) with probability \(p(x, y)\). For the matrix \(A\) we get:

\[
b(1) = \frac{1}{2} + \frac{1}{2}(b(2) + b(3))
\]
The solution of this system of equations is \( b = (4, 4, 3) \)
as before.

\((M/M/1 \ queue)\).

Suppose that there is a queue with one server. People get into the line at a rate of \( \lambda \) and they get served at the rate of \( \mu \).

1. This is a continuous Markov chain \( X_t \) with states \( 0, 1, 2, 3, \ldots \). What is the infinitesimal generator \( A = (a(x, y)) \)?
   This is given by
   \[
   a(n, n + 1) = \lambda \quad \text{for} \quad n \geq 0
   \]
   \[
   a(n, n - 1) = \mu \quad \text{for} \quad n > 0
   \]
   and don’t forget:
   \[
   a(n, n) = -\lambda - \mu \quad \text{for} \quad n \geq 0
   \]
   The other entries of \( A \) are all zero.

2. Convert this to a countable Markov chain \( Z_n \). What is the (infinite) probability transition matrix \( P = (p(x, y)) \)?
   \[
   p(n, n + 1) = \frac{\lambda}{\lambda + \mu} \quad \text{for} \quad n > 0
   \]
   \[
   p(n, n - 1) = \frac{\mu}{\lambda + \mu} \quad \text{for} \quad n > 0
   \]
   \[
   p(0, 1) = 1
   \]
   and the other entries of \( P \) are zero. These are the probabilities for the jumps between states.
   \[
   p(x, y) = \mathbb{P}(\text{the state jumps to} \ y \ \text{from} \ x)
   \]

3. Using your answers to Homework Problem #2.1 determine
   (a) Under what conditions is this queue transient, positive recurrent, null recurrent?

   This is a little tricky. The conversion is
   \[
   p = \frac{\lambda}{1 + \lambda} = 1 - \frac{1}{1 + \lambda}
   \]
   \[
   q = \frac{\mu}{1 + \mu} = 1 - \frac{1}{1 + \mu}
   \]
   Since the function \( f(x) = 1 - 1/(1 + x) \) has positive derivative:
   \[
   f'(x) = \frac{1}{(1 + x)^2} > 0
   \]
   So, \( f(x) \) is monotonically increasing. I.e., \( p > q \iff \lambda > \mu \) and \( p < q \iff \lambda < \mu \). So this countable Markov chain is
   (i) transient iff \( p > q \iff \lambda > \mu \)
   (ii) null recurrent iff \( p = q \iff \lambda = \mu \)
   (iii) positive recurrent iff \( p < q \iff \lambda < \mu \)
To prove this you have to also convert 2.1 into jump probabilities.

\[ p(x, x + 1) = \mathbb{P}(\text{the state jumps to } x + 1 \text{ from } x) = \frac{p(1 - q)}{p(1 - q) + q(1 - p)} = \frac{\lambda}{\lambda + \mu} \]

\[ p(x, x - 1) = \mathbb{P}(\text{the state jumps to } x - 1 \text{ from } x) = \frac{q(1 - p)}{p(1 - q) + q(1 - p)} = \frac{\mu}{\lambda + \mu} \]

(b) When it is positive recurrent, what is the expected return time to 0? (For \( X_t \) not \( Z_n \)).

The invariant distribution (for \( Z_n \), not for \( X_n \)) is given by normalizing the sequence:

\[
\left( \frac{\lambda}{\lambda + \mu}, \frac{\lambda}{\mu}, \frac{\lambda^2}{\mu^2}, \ldots \right)
\]

The sum of these numbers is

\[
\frac{\lambda}{\lambda + \mu} + \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{2\lambda\mu}{\mu^2 - \lambda^2}
\]

So,

\[
\pi(0) = \frac{\mu^2 - \lambda^2}{2\lambda\mu} \frac{\lambda}{\lambda + \mu} = \frac{\mu - \lambda}{2\mu}
\]

So, the expected return time to 0 for \( Z_n \) is

\[
\frac{1}{\pi(0)} = \frac{2\mu}{\mu - \lambda} = 1 + \frac{\lambda + \mu}{\mu - \lambda}
\]

(Note that this number is \( \geq 2 \) since it takes at least 2 steps to go from 0 back to 0.) For the continuous chain \( X_t \), the first step takes \( 1/\lambda \) amount of time and every subsequent step takes \( 1/(\lambda + \mu) \) amount of time. So the correct answer is:

\[
\mathbb{E}(T) = \frac{1}{\lambda} + \left( \frac{\lambda + \mu}{\mu - \lambda} \right) \left( \frac{1}{\mu + \lambda} \right) = \frac{1}{\lambda} + \frac{1}{\mu - \lambda}
\]

(I wonder if anyone got this.)

(4) Determine when the chain \( X_t \) is explosive.

This chain is never explosive for the simple reason that each jump takes an average of

\[
\frac{1}{\lambda + \mu}
\]

amount of time. So, you can’t have an infinite number of jumps in a finite amount of time. In particular, you can’t go to infinity. Another method is to show that the amount of time it takes to get from state \( n \) to state \( n + 1 \) is bounded below by \( \frac{1}{\lambda + \mu} \). So, the infinite sum diverges and you have no explosion.