

**MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK**

HOMEWORK 4: OPTIMAL STOPPING TIME

Two problems due Wednesday, March 19. Answers will be posted the following week.

Quiz 2 postponed until Thursday, March 20.

4.6 with additional questions.

You toss a die and your payoff is

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 & \text{otherwise} \end{cases}$$

(a) [What is the optimal strategy and what is your expected winnings if you use the optimal strategy?](#)

The algorithm is to find a descending sequence of superharmonics u_1, u_2, \dots converging to $v(x)$, then determine the strategy from this decimal approximation to $v(x)$ and then from that to determine the precise formula for $v(x)$. Since this problem is nonstochastic, the value function is determined by its expected value

$$E = \sum p_x v(x)$$

We start with the very optimistic function:

$$u_1 = (0, 36, 36, 36, 36, 36)$$

This means, if you don't lose on the first roll, you assume that you will get the highest possible payoff (36). The average of these numbers is

$$E_1 = (5/6)(36) = 30.$$

$$u_2(x) = \max(f(x), E_1) = (0, 30, 30, 30, 30, 36)$$

with average

$$E_2 = 26$$

$$u_3 = (0, 26, 26, 26, 26, 36)$$

$$E_3 = 23.33333333$$

$$E_4 = 21.83333333$$

$$E_5 = 21.08333333$$

...

$$E_{25} = 20.33333369$$

So, the winning strategy is to stop at 5 or 6 and play at 2,3,4. The value function is

$$v(0, E, E, E, 25, 36)$$

with average

$$E = \frac{3E + 25 + 36}{6} = \frac{1}{2}E + \frac{61}{6}$$

So,

$$E = \frac{61}{3} = 20\frac{1}{3}$$

$$v = (0, 20\frac{1}{3}, 20\frac{1}{3}, 20\frac{1}{3}, 25, 36)$$

The second question is a little ambiguous since it is not clear exactly what “expected winnings” means. The expected winning of this game is $E = 20\frac{1}{3}$ before you roll the first die. After you roll and get x your expected winning is given by $v(x)$ which depends on x .

(b) Suppose that there is a cost of $g(x) = r$. What is the smallest value of r so that the optimal strategy is to stop at any number except 2?

Since you know the optimal strategy, you can skip the first steps and go to the last step. The value function is:

$$v = (0, E - r, 9, 16, 25, 36)$$

But $E - r$ must be ≤ 9 : otherwise you should play at $x = 3$. So

$$r \geq E - 9$$

The smallest value of r is when these are equal. So

$$v = (0, 9, 9, 16, 25, 36)$$

with average

$$E = \frac{95}{6} = 15\frac{5}{6}$$

which makes

$$r = E - 9 = 6\frac{5}{6}$$

(c) Suppose there is a discount α but no cost. What is the largest value of α so that you should stop no matter what you get?

Again, you know what the strategy is. So,

$$v = (0, 4, 9, 16, 25, 36)$$

with average

$$E = 90/6 = 15$$

This means that

$$v(2) = \max(f(2), \alpha E) = \max(4, 15\alpha) = 4$$

The largest that α could be is $4/15$.

(d) Suppose that $g(x) = 5$ and $\alpha = .8$ Then what is the optimal strategy? Calculate the value function $v(x)$.

Hint: The iteration algorithm should start with:

$$u_1(x) = \begin{cases} 0 & \text{if recurrent} \\ f(x) & \text{if } f(x) \geq \max(\alpha f(y) - g(x)) \\ \max(\alpha f(y) - g(x)) & \text{otherwise} \end{cases}$$

This gives

$$u_1 = (0, 23.8, 23.8, 23.8, 25, 36)$$

with average

$$\begin{aligned}
 E_1 &= 22.0666666\dots \\
 u_2(x) &= \max(f(x), \alpha E_1 - g(x)) \\
 u_2 &= (0, 12.65333333, 12.65333333, 16, 25, 36) \\
 E_2 &= 17.05111111 \\
 E_3 &= 15.77348148 \\
 E_4 &= 15.60313086 \\
 &\dots \\
 E_{25} &= 15.57692308
 \end{aligned}$$

with

$$u_{25} = (0, 7.461538462, 9, 16, 25, 36)$$

So, you should continue with 2. Your value function is

$$v = (0, .8E - 5, 9, 16, 25, 36)$$

with average

$$\begin{aligned}
 E &= \frac{1}{6}(.8E - 5 + 9 + 16 + 25 + 36) \\
 6E &= .8E + 81 \\
 E &= 810/52 = 15.57692308 \\
 .8E - 5 &= 97/13 = 7.461538462
 \end{aligned}$$

which agrees with E_{25} and $u_{25}(2)$. So,

$$v = (0, 97/13, 9, 16, 25, 36).$$

Second problem: This is random walk with absorbing wall on the left and reflecting wall on the right.

$$\begin{array}{rcccccccc}
 x = & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 f(x) = & 0 & 1 & 2 & 5 & 6 & 21 & 19 \\
 g(x) = & 22 & 2 & 2 & 2 & 1 & 1 & 1
 \end{array}$$

a) Draw a graph of $f(x)$ and connect the dots.

b) Find the optimal strategy and value function $v(x)$ if there is no cost. Describe geometrically what is the algorithm when the right wall is reflecting. [Hint: consider the mirror image of the function on the reflecting wall.]

The optimal strategy is to stay at 5 and play at any other point (except of course 0 where you lose). This is because the value function is the convex of the function $f(x)$ and its mirror image:

$$v = (0, 21/5, 42/5, 63/5, 84/5, 21, 21)$$

c) Find the optimal strategy and value function if the cost $g(x)$ is given as above.

According to the book, you should start with $u_1 =$ a superharmonic, such as $v(x)$ from part b.

$$u_1 = (0, 21/5, 42/5, 63/5, 84/5, 21, 21)$$

Then iterate:

$$u(n+1)(x) = \max\left(f(x), \frac{u_n(x-1) + u_n(x)}{2} - g(x)\right)$$

where $u_n(7) = u_n(5)$. The limit is reached (numerically) at $n = 25$:

$$v = u_{25} = (0, 1, 2, 5, \mathbf{12}, 21, \mathbf{20})$$

If we compare this with the average of the neighbors minus the cost we get:

$$(0, -1, 1, 5, 12, 15, 20)$$

The maximum of these number and the payoff $f(x)$ is $v(x)$. So, $v(x)$ is the value function.

The strategy is to play at $x = 4$ and $x = 6$ and stop at any other number.

d) Find the optimal strategy and value function if there is a discount $\alpha = .9$ and no cost.

Start with the same u_1 and iterate:

$$u_{n+1}(x) = \max \left(f(x), \alpha \frac{u_n(x-1) + u_n(x+1)}{2} \right)$$

to get

$$u_{25} = (0, 1.987787978, 4.416562181, 7.825305772, 12.97180168, 21, 19)$$

We see that the strategy is to play at $x = 1, 2, 3, 4$ and stop at $x = 5, 6$.

If you want the exact value of $v(x)$ you have to solve the system of linear equations:

$$v(1) = \frac{9}{20}v(2)$$

$$v(2) = \frac{9}{20}(v(1) + v(3))$$

$$v(3) = \frac{9}{20}(v(2) + v(4))$$

$$v(4) = \frac{9}{20}(v(3) + 21)$$

e) Find the optimal strategy and value function if there is a discount $\alpha = .9$ and cost $g(x)$ as given above.

$$u_{n+1}(x) = \max \left(f(x), \alpha \frac{u_n(x-1) + u_n(x+1)}{2} - g(x) \right)$$

Note that $g(x)$ is not multiplied by α because the interpretation is that you pay first (in today's dollars) and your payoff is in tomorrow's dollars. The limit is:

$$u_{25} = (0, 1, 2, 5, 10.7, 21, 19)$$

This implies that you should play at $x = 4$ and stop everywhere else. The exact value of $v(4)$ is

$$v(4) = \alpha \frac{f(3) + f(5)}{2} - g(4) = \frac{9}{20}(5 + 21) - 1 = 10.7$$