MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK 2006

Homework 5
Martingales

These problems were due Wednesday, March 26.

First problem: Suppose that $X_1, X_2, \cdots$ are i.i.d. random variables with expected value $\mu = \mathbb{E}(X_i)$.

a) Show that

$$M_n = \frac{X_1 X_2 \cdots X_n}{\mu^n}$$

is a martingale wrt $X_1, X_2, \cdots$.

$$M_{n+1} = M_n \frac{X_{n+1}}{\mu}$$

So,

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = M_n \frac{\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n)}{\mu} = M_n \frac{\mu}{\mu} = M_n$$

b) Assuming that $X_i > 0$, is $\ln M_n$ (natural log of M_n) a martingale in general? [See next question.] No, you need to subtract its expected value.

c) Modify $\ln M_n$ to make it into a martingale.

Just let

$$\alpha := \mathbb{E}(\ln \frac{X_i}{\mu}) = \mathbb{E}(\ln X_i) - \ln \mu$$

then

$$\ln M_n - n\alpha$$

is a martingale.

Second problem: (a special case of the first problem)

d) Suppose that (in the above problem)

$$X_i = \begin{cases} 1/2 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2 \end{cases}$$

Then show that $T =$ first time that $M_n = 1/2$ is a stopping time. This is obviously true. However, one student pointed out that, since $\mu = 5/4$ it is impossible for M_n to be equal to $1/2$. So, you could also say that this is not a stopping time.

Does the Optimal Sampling Theorem hold? OST says that

$$\mathbb{E}(M_T) = M_0$$

but these numbers are $1/2$ and 1 which are not equal.

e) Using the Central Limit Theorem (on $\ln M_n$), show that, with probability one, M_n converges to 0.

Since

$$\ln(X/\mu) = \begin{cases} -\ln 2 - \ln(5/4) & \text{with probability } 1/2 \\ +\ln 2 - \ln(5/4) & \text{with probability } 1/2 \end{cases}$$

the expected value is

$$\mathbb{E}(\ln(X/\mu)) = -\ln(5/4) = -0.223$$

It is a negative constant. The standard deviation of $\ln(X/\mu)$ is the square root of the expected square deviation which is

$$\sigma = \ln 2 = 0.693$$

The central limit theorem says that as n goes to infinity,

$$\ln M_n \sim N(-0.223n, 0.693\sqrt{n})$$

Since the expected value

$$\mathbb{E}(\ln M_n) = \mu_n = -0.223n$$

is a large number of standard deviations

$$\sigma_n = \sigma(\ln M_n) = 0.693\sqrt{n}$$

below a number which is going to $-\infty$. (For example,

$$\mu_n = -0.223n = -0.1n - 0.178\sqrt{n}(\sigma_n)$$

it means that $\ln M_n < -0.1n$ with probability going to 1 as $n \to \infty$. So, $\ln M_n$ goes to $-\infty$ which means that $M_n \to 0$.)

Explain why this contradicts the Martingale convergence theorem. (What part of the statement holds? What part doesn't hold?)

The convergence theorem says, first, M_n converges to some limit M_∞. This is true since M_n converges to 0. The second statement is that

$$\mathbb{E}(M_n) \to \mathbb{E}(M_\infty)$$

This part is not true since $\mathbb{E}(M_n) = 1$ and $\mathbb{E}(M_\infty) = 0$.

Third problem Suppose that

$$X_i = \begin{cases} -1 & \text{with probability } 2/3 \\ 2 & \text{with probability } 1/3 \end{cases}$$

f) Show that

$$S_n = X_1 + X_2 + \cdots + X_n$$

is a martingale wrt $X_1, X_2, \cdots$

Since each X_i has $\mathbb{E}(X_i) = 0$,

$$\mathbb{E}(S_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1} \,|\, \mathcal{F}_n) = S_n + \mathbb{E}(X_{n+1}) = S_n.$$

So, S_n is a martingale.

g) Show that the Optimal Sampling Theorem does not hold. [Let T = first time that $S_n > 0$. Show that S_n is a recurrent Markov Chain and therefore $\mathbb{P}(T < \infty) = 1$]

To test if the process is recurrent we test it for transience. We look for a function $a(x)$ so that $a(0) = 1$, $\inf a(n) = 0$ and

$$a(n) = \sum p(n,m)a(m)$$

This is:

$$a(n) = \frac{2}{3}a(n-1) + \frac{1}{3}a(n+2)$$

To solve this you put $a(n) = c^n$. Then:

$$c^n = \frac{2}{3}c^{n-1} + \frac{1}{3}c^{n+2}$$

Multiply by 3 and divide by c^{n-1} to get:

$$3c = 2 + c^3$$

or:

$$c^3 - 3c + 1 = 0$$

This factors as:

$$(c-1)^2(c+2) = 0$$

So, the roots are $c = 1, -2$. Since there are not positive roots aside from 1, this implies that the Markov chain is not transient. More precisely, the homogeneous solutions are $1, n, (-2)^n$. So the general solution is:

$$a(n) = A + nB + (-2)^n C$$

where A, B, C are constants adding up to 1. But $a(n)$ is a probability. So, $B = C = 0$ (otherwise we get a number greater than 1). So,

$$a(n) = 1.$$

So, there are no solutions with $\inf a(n) = 0$. So, the chain is not transient. So, it must be recurrent.

This means $T < \infty$ with probability 1. (The chain is actually null recurrent but this only means that the expected value of T is infinite. This doesn't matter.)

The OST says that

$$\mathbb{E}(M_T) = M_0$$

if T is a stopping time with $\mathbb{P}(T < \infty)$. This is not true in this case since $M_T = 1$ and $M_0 = 0$. (This means of course that M_n cannot be uniformly integrable.)

h) Is $S_n/\sqrt{n}$ a martingale wrt $X_1, X_2, \cdots$?

Let $M_n = S_n/\sqrt{n}$. Then

$$M_{n+1} = \frac{S_{n+1}}{\sqrt{n+1}} = \frac{S_n + X_{n+1}}{\sqrt{n+1}} = \frac{S_n}{\sqrt{n+1}} + \frac{X_{n+1}}{\sqrt{n+1}}$$
$$= M_n\frac{\sqrt{n}}{\sqrt{n+1}} + \frac{X_{n+1}}{\sqrt{n+1}}$$

Since $\mathbb{E}(X_{n+1}) = 0$, this gives:

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = M_n\frac{\sqrt{n}}{\sqrt{n+1}} \neq M_n.$$

So, this is not a martingale.