

MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK

HOMEWORK 6
RENEWAL

These problems are due Thursday, April 3. Answers will be posted the following week.

First problem: Suppose that we have a renewal process with a uniform distribution

$$f(t) = \begin{cases} 1/10 & \text{if } 0 < t \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(i.e., we have a light bulb which lasts at most 10 days and can burn out at any time with equal probability during those 10 days)

This means that $F(t) = t/10$ for $0 \leq t \leq 10$.

$$F(t) = \begin{cases} t/10 & \text{if } 0 < t \leq 10 \\ 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 10 \end{cases}$$

You walk into the warehouse after one year. The light bulbs are distributed according to the equilibrium distribution.

- (1) What is the average age of a light bulb (in this equilibrium distribution)?
 $t = 1\text{year}$. The age of the light bulb is A_t . The density function for A_t is

$$f_A(s) = \frac{1 - F(s)}{\mu} = \frac{1 - s/10}{5} = \frac{1}{5} - \frac{s}{50}$$

for $0 \leq s \leq 10$. So, the average value of A is

$$\mathbb{E}(E_t) = \int_0^{10} \left(\frac{s}{5} - \frac{s^2}{50} \right) ds = 10 - \frac{1000}{150} = \frac{10}{3}$$

- (2) If you pick a light bulb and it has been burning for 4 days, what is the probability that it will burn for at least 4 more days.

I decided this is not a good question because naïve methods give the correct answer whereas sophisticated methods give the wrong answer. The correct answer is $1/3$.

The naïve explanation is that this is the conditional probability that a light bulb will last for at least 8 days given that it has lasted for 4 days. This is

$$\mathbb{P}(T \geq 8 | T \geq 4) = \frac{\mathbb{P}(T \geq 8)}{\mathbb{P}(T \geq 4)} = \frac{2/10}{6/10} = \frac{1}{3}$$

But his calculation does not use the equilibrium distribution. A more sophisticated interpretation is that this is the conditional probability that $C \geq 8$ given that $C \geq 4$:

$$\mathbb{P}(C \geq 8 | C \geq 4) = \frac{\mathbb{P}(C \geq 8)}{\mathbb{P}(C \geq 4)} =$$

But:

$$\mathbb{P}(C \geq t) = \int_t^{10} \frac{xf(x)}{\mu} dx = \int_t^{10} \frac{x}{50} dx = 1 - \frac{t^2}{100}$$

So,

$$\mathbb{P}(C \geq 8 | C \geq 4) = \frac{\mathbb{P}(C \geq 8)}{\mathbb{P}(C \geq 4)} = \frac{1 - \frac{64}{100}}{1 - \frac{16}{100}} = \frac{3}{7}$$

However, this second calculation does not take into account that the condition is not just that $C \geq 4$ but also that you observed the light bulb when it was 4 days old. So, what I think is a very careful answer is:

$$\mathbb{P}(C \geq 8 | 4 < A \leq 4 + dt) = \frac{\mathbb{P}(C \geq 8 \text{ and } 4 < A \leq 4 + ds)}{\mathbb{P}(4 < A \leq 4 + ds)}$$

The probability that the age is 4 is the density function of A

$$\mathbb{P}(4 < A \leq 4 + ds) = f_A(4)ds = \frac{1 - F(4)}{\mu} ds = \frac{1 - 4/10}{5} ds = \frac{3}{25} ds$$

$$\mathbb{P}(C \geq 8) = \int_8^{10} f_C(x) dx$$

Given the total life, the probability that you observe the light bulb when it is exactly 4 days old is ds/C . So,

$$\mathbb{P}(C \geq 8 | 4 < A \leq 4 + dt) = \int_8^{10} \frac{f_C(x) ds dx}{x} = \int_8^{10} \frac{xf(x)}{\mu} \frac{ds dx}{x} = \int_8^{10} \frac{1/10}{5} ds dx = \frac{1}{25} ds$$

Which gives

$$\mathbb{P}(C \geq 8 | 4 < A \leq 4 + dt) = \frac{ds/25}{3ds/25} = \frac{1}{3}$$

Second problem: a) Calculate the probability distribution of $Z = X + Y$ if X, Y are exponential variables with rate 2,3 resp.

The density function of Z is the convolution $f_Z = f_X * f_Y$:

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(x) f_Y(z-x) dx \\ &= \int_0^z 2e^{-2x} 3e^{-3(z-x)} dx = 6 \int_0^z e^{-3z+x} dx = 6e^{-3z+x} \Big|_0^z = 6e^{-2z} - 6e^{-3z} \end{aligned}$$

b) Calculate the probability distribution of $W = X - Y$. [Hint: $-Y$ is a random variable with density function $f(y) = 3e^{3y}$ for $y < 0$ and $f(y) = 0$ if $y \geq 0$.

Using the hint: $f_W = f_X * f_{-Y}$ which is given by:

$$f_W(w) = \int f_X(x) f_{-Y}(w-x) dx$$

The region of integration is given by $x > 0$ and $y = w - x < 0$ which means $x > w$. So, $x > \max(0, w)$. This makes the region of integration $[w, \infty)$ if $w \geq 0$ and $[0, \infty)$ if $w < 0$. So, for $w \geq 0$ we get:

$$f_W(w) = \int_w^{\infty} 2e^{-2x} 3e^{3(w-x)} dx = \frac{6}{-5} e^{3w-5x} \Big|_w^{\infty} = \frac{6}{5} e^{-2w}$$

If $w < 0$ then

$$f_W(w) = \int_0^\infty 2e^{-2x} 3e^{3(w-x)} dx = \frac{6}{-5} e^{3w-5x} \Big|_{-\infty}^0 = \frac{6}{5} e^{3w}$$

So, the final answer is:

$$f_W(w) = \begin{cases} \frac{6}{5} e^{-2w} & \text{if } w \geq 0 \\ \frac{6}{5} e^{3w} & \text{if } w < 0 \end{cases}$$

Third problem: We have a water tower which gives water to a desert town. It has been empty for a few day. :(

Suppose that it showers from time to time. This event is a Poisson process with rate $\lambda = 1/7$ (once a week on average). When it showers it always dumps 5,000 gallons of water into the tank. The residents of this town use 1,000 gallons of water per day continuously (at a constant rate throughout the day when there is water).

Convert this into an $M/G/1$ queue

The water in the tank is the queue. The residents together form the server which “serves” the inhabitants of the queue by drinking it. The water must be counted in units of 5,000 gallons. Then each rainfall is one unit of water. The rainfalls are the people in line. It is given that these units of water enter the queue at the rate of $\lambda = 1/7$.

and determine:

- (1) What is the meaning of X_n, Y_n, U_n, τ in this case?

X_n is the number of times it rains during the time that the n th rainfall of water is being used. This is the number of times it rains between $5n$ days and $5n + 5$ days after the first rain.

Y_n is the number of units of water in the tower $5n$ days after the first rain.

U_n is 5 days for all n .

τ is the number of the number of times it rains before the tower is empty. Another way to say it is that 5τ is the total length of time that there was water in the tower during one period of time that there was water in the tower.

- (2) Compute μ .

$\mu = \mathbb{E}(U_n) = 5$ because it is always 5.

- (3) If there is a shower tomorrow, how long can the town expect to have water?

This is the expected length of the first service period

$$\mathbb{E}(S_1) = \mathbb{E}(\tau)\mu = \frac{\mu}{1 - \lambda\mu} = \frac{5}{1 - 5/7} = \frac{35}{2} = 17\frac{1}{2}$$

The town can expect to have rain for 17 and a half days.