

**MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK**

HOMEWORK 8
BROWNIAN MOTION

Three problems are due on the last day of class. Answers will be posted the following week.

First problem: (reflection principle) Let W_t be standard Brownian motion. Calculate the probability that there exist $0 < a < b < c < 1$ so that $W_a = 1, W_b = -1, W_c = 0$. Give the details of the argument.

Let P be this probability:

$$P = \mathbb{P}(W_a = 1, W_b = -1, W_c = 0 \text{ for some } 0 < a < b < c < 1).$$

Then the final answer is:

$$P = 2(1 - \Phi(4)) \doteq 6.3372 \times 10^{-5} = 0.000063372$$

where $\Phi(x)$ is the CDF of the standard normal variable:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx$$

Here is the proof using the reflection principle.

First, you need to realize that, after reaching 0 at $t = c$ we either go up or down with equal probability. So, by the reflection principle we have:

$$P = 2\mathbb{P}(W_a = 1, W_b = -1 \text{ for some } 0 < a < b < 1 \text{ and } W_1 > 0)$$

To get from $W_b = -1$ to $W_1 > 0$ has the same probability as going from -1 to below -2 . So,

$$P = 2\mathbb{P}(W_a = 1 \text{ for some } 0 < a < 1 \text{ and } W_1 < -2)$$

Going from $W_a = 1$ to $W_1 < -2$ has the same probability as going from 1 to 4. So, this is:

$$P = 2\mathbb{P}(W_1 > 4)$$

But, W_1 is standard normal. So, this is

$$P = 2(1 - \Phi(4)) = 2 - 2\Phi(4).$$

Use scaling to find the probability that there exist $0 < a < b < c < t$ so that $W_a = 1, W_b = -1, W_c = 0$.

We want:

$$\mathbb{P}(W_a = 1, W_b = -1, W_c = 0 \text{ for some } 0 < a < b < c < t)$$

By the same argument as before, this is equal to

$$2\mathbb{P}(W_t > 4)$$

But $W_t \sim N(0, t)$. So, the probability that $W_t > 4$ is:

$$1 - \Phi(4/\sqrt{t})$$

So, the answer is:

$$\mathbb{P}(W_a = 1, W_b = -1, W_c = 0 \text{ for some } 0 < a < b < c < t) = 2 - 2\Phi\left(\frac{4}{\sqrt{t}}\right)$$

Second problem: (fractal dimension) Take the unit interval $[0, 1]$ and remove the open sets $(1/4, 3/8)$ and $(5/8, 3/4)$. In other words, you remove two open intervals $1/8$ of a unit long leaving three closed intervals $1/4$ of a unit long. Repeat the process infinitely often. Each time you remove two open pieces from each interval that you have leaving three closed intervals which are exactly $1/4$ the size of the interval. Find the fractal dimension of the resulting set using the two methods taught in class:

- (1) **By scaling.** (This is the method we used to compute the fractal dimension of the Cantor set which is very similar to this set.)

Call this set X . If we scale the set X up by a factor of 4 then there will be 3 sets which look just like X . So, the dimension D of X satisfies:

$$3 = 4^D$$

In other words,

$$D = \frac{\ln 3}{\ln 4}.$$

- (2) **By cutting up the interval into smaller intervals and counting how many intervals we need to cover the set.** (This is the definition of the box dimension.)

The set X can be covered by 3 intervals of length $1/4$ since

$$X \subset [0, 1/4] \cup [3/8, 5/8] \cup [3/4, 1]$$

I claim that X requires 3^k intervals of length $1/4^k$ to cover it. This is proved by induction on k . [Actually the problem does not say to “prove” it. But you need to give an explanation.] The statement is true for $k = 1$ since the three points $0, 1, 3/8 \in X$ need to be covered by different intervals of $1/4$. If you know that you need 3^k intervals of length $1/4^k$ then, by scaling down by a factor of 4 we see that each of the three pieces:

$$X \cap [0, 1/4], \quad X \cap [3/8, 5/8], \quad X \cap [3/4, 1]$$

needs 3^k intervals of length $1/4^{k+1}$ since they are the same as X but scaled down by a factor of 4. So, we need 3 times that many intervals to cover all of X . So, we need 3^{k+1} intervals of length $1/4^{k+1}$ to cover X . This proves the claim by induction on k . The dimension D of X is given by

$$3^k = (4^k)^D$$

This gives

$$D = \frac{k \ln 3}{k \ln 4} = \frac{\ln 3}{\ln 4}$$

which is the same as before.

Note: Your two answers should be equal.

Third problem: (heat equation) Suppose that B is the infinite horizontal strip:

$$B = \{(x, y) \in \mathbb{R}^2 \mid |y| < 1\}$$

Let g be the function on the boundary of B given by

$$g(x, \pm 1) = x$$

- (1) Solve the heat equation $\Delta f = 0$. [This is very easy. Just guess. Use the fact that the solution is unique.]

$f(x, y) = x$ is the unique solution.

- (2) Write the solution as a probability statement. $\mathbb{E}^x(\dots) = \dots$

The probability statement is: Starting at the point $X_0 = (x, y) \in B$, a particle moves according to 2-dimensional Brownian motion. If T is the first time that the particle hits the boundary of B then T is the stopping time. Then $f(x, y)$ is the expected value of X_T :

$$f(x, y) = \mathbb{E}(g(X_T) \mid X_0 = (x, y))$$

- (3) Give a probabilistic argument to prove this. [This uses a reflection type argument on an unknown density function.]

Draw a vertical line through the point (x, y) . Paths going to the right of this line and those going to the left are mirror images and therefore have equal probability. So, the point X_T is equally likely to lie in the intervals $[x + s, x + s + ds] \times \pm 1$ as it is to lie in the interval $[x - s - ds, x - s] \times \pm 1$. So, the average value of the first coordinate will be x .