MATH 56A: FALL 2006
HOMEWORK AND ANSWERS

1. Homework 1

1.1. **Linear diffeq's and recursions.** Three problems:

0.1 Find all functions $x(t), y(t)$ so that

$$x'(t) = -x + y, \quad y'(t) = 3x - 3y$$

Find the particular solution so that $x(0) = y(0) = 1/2$.

0.5 Find all functions f from integers to real numbers so that

$$f(n) = \frac{1}{2}f(n+1) + \frac{1}{2}f(n-1) - 1$$

[Show first that $f(n) = n^2$ is a particular solution.]

0.6 (a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f''(x) + f'(x) + f(x) = 0$$

(b) Final all functions $f : \mathbb{Z} \to \mathbb{R}$ so that

$$f(n+2) + f(n+1) + f(n) = 0$$

1. Homework 1 answers

1.1. **Linear diffeq's and recursions.** four answers:

0.1 Find all functions $x(t), y(t)$ so that

$$x'(t) = -x + y, y'(t) = 3x - 3y$$

Find the particular solution so that $x(0) = y(0) = 1/2$.

The matrix is

$$A = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

This has eigenvalues $0, -4$ with corresponding eigenvectors $X_1 = \binom{1}{1}, X_2 = \binom{1}{-3}$. So $A = QDQ^{-1}$ where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}, \quad Q^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

And

$$e^{tA} = Qe^{tD}Q^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-4t} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3+e^{-4t} & 1-e^{-4t} \\ 3-3e^{-4t} & 1+3e^{-4t} \end{pmatrix}$$

The general solution is $X = e^{tA}X_0$ or

$$x = \frac{x_0}{4}\left(3 + e^{-4t}\right) + \frac{y_0}{4}\left(1 - e^{-4t}\right)$$

$$y = \frac{x_0}{4}\left(3 - 3e^{-4t}\right) + \frac{y_0}{4}\left(1 + 3e^{-4t}\right)$$

When $x_0 = y_0$ then the e^{-4t} terms all cancel and we get that x, y are constant functions. In particular, $x = y = \frac{1}{2}$ is the particular solution in the homework.

Some people found another method but didn't carry it through: Some of you noticed that the equations say: $y' = -3x'$ or

$$\frac{dy}{dt} = -3\frac{dx}{dt}$$

Cancel the dt's and integrate:

$$\int dy = \int -3dx$$

which gives: $y = -3x + C_1$. Now you have to continue and put it back into the original equation

$$\frac{dx}{dt} = -x + y = -x - 3x + C_1 = -4x + C_1$$

$$\frac{dx}{-4x + C_1} = dt$$

$$-\frac{1}{4}\ln|-4x + C_1| = t + C_2$$

$$-4x + C_1 = \pm e^{-4t-4C_2}$$

So,

$$x = C_3 e^{-4t} + \frac{1}{4}C_1$$

and

$$y = -3x + C_1 = -3C_3 e^{-4t} + \frac{1}{4}C_1$$

When you put in the initial conditions you find $C_1 = 2, C_3 = 0$.

Remember that you need to add $+C$ with a new C every time you integrate.

0.5 Find all functions f from integers to real numbers so that

$$f(n) = \frac{1}{2}f(n+1) + \frac{1}{2}f(n-1) - 1$$

[Show first that $f(n) = n^2$ is a particular solution.]

To solve the homogeneous equation, try $f = a^n$. If a is a double root then the second solution is $f(n) = na^n$. The homogenous equation gives

$$a^2 - 2a + 1 = 0$$

This has only one root: $a = 1$. So, the solutions are $f(n) = 1$ and $f(n) = n$. Thus the general solution is

$$n^2 + bn + c$$

where b and c are constants. There is no constant in front of the particular solution.

0.6 (a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f''(x) + f'(x) + f(x) = 0$$

Here you try $f(x) = e^{\lambda x}$ and you find that

$$\lambda^2 + \lambda + 1 = 0$$

or

$$\lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

$$e^{\lambda x} = e^{-x/2}(\cos\frac{\sqrt{3}x}{2} \pm i\sin\frac{\sqrt{3}x}{2})$$

To get a real solution students correctly took a linear combination of the real and imaginary parts:

$$f(x) = ae^{-x/2}\cos\frac{\sqrt{3}x}{2} + be^{-x/2}\sin\frac{\sqrt{3}x}{2}$$

(b) Find all functions $f : \mathbb{Z} \to \mathbb{R}$ so that

$$f(n+2) + f(n+1) + f(n) = 0$$

You try $f(n) = a^n$ and you get

$$a^2 + a + 1 = 0$$

Or

$$a = \frac{-1 \pm i\sqrt{3}}{2}$$

So, an arbitrary complex solution is given by

$$f(n) = a\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + b\left(\frac{-1 - i\sqrt{3}}{2}\right)^n$$

In order for this to be a real number it must be equal to its complex conjugate. So, $b = \overline{a}$. I.e., $a = c + id, b = c - id$ where

$$c = f(0)/2$$
$$d = -f(1)\sqrt{3}/3$$