1. Homework 1

1.1. Linear diffeq’s and recursions. Three problems:

0.1 Find all functions $x(t), y(t)$ so that

$$x'(t) = -x + y, \quad y'(t) = 3x - 3y$$

Find the particular solution so that $x(0) = y(0) = 1/2$.

0.5 Find all functions $f$ from integers to real numbers so that

$$f(n) = \frac{1}{2}f(n + 1) + \frac{1}{2}f(n - 1) - 1$$

[Show first that $f(n) = n^2$ is a particular solution.]

0.6 (a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f''(x) + f'(x) + f(x) = 0$$

(b) Find all functions $f : \mathbb{Z} \to \mathbb{R}$ so that

$$f(n + 2) + f(n + 1) + f(n) = 0$$
1. **Homework 1 answers**

1.1. **Linear diffeq’s and recursions.** four answers:

0.1 Find all functions \(x(t), y(t)\) so that

\[
x'(t) = -x + y, \quad y'(t) = 3x - 3y
\]

Find the particular solution so that \(x(0) = y(0) = 1/2\).

The matrix is

\[
A = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}
\]

This has eigenvalues 0, -4 with corresponding eigenvectors \(X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\), \(X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}\). So \(A = QDQ^{-1}\) where

\[
Q = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}, \quad Q^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}
\]

And

\[
e^{tA} = Qe^{tD}Q^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-4t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 + e^{-4t} & 1 - e^{-4t} \\ 3 - 3e^{-4t} & 1 + 3e^{-4t} \end{pmatrix}
\]

The general solution is \(X = e^{tA}X_0\) or

\[
x = \frac{x_0}{4} (3 + e^{-4t}) + \frac{y_0}{4} (1 - e^{-4t})
\]

\[
y = \frac{x_0}{4} (3 - 3e^{-4t}) + \frac{y_0}{4} (1 + 3e^{-4t})
\]

When \(x_0 = y_0\) then the \(e^{-4t}\) terms all cancel and we get that \(x, y\) are constant functions. In particular, \(x = y = \frac{1}{2}\) is the particular solution in the homework.

Some people found another method but didn’t carry it through: Some of you noticed that the equations say: \(y' = -3x'\) or

\[
\frac{dy}{dt} = -3 \frac{dx}{dt}
\]

Cancel the \(dt\)'s and integrate:

\[
\int dy = \int -3dx
\]

which gives: \(y = -3x + C_1\). Now you have to continue and put it back into the original equation

\[
\frac{dx}{dt} = -x + y = -x - 3x + C_1 = -4x + C_1
\]

\[
\frac{dx}{-4x + C_1} = dt
\]

\[
-\frac{1}{4} \ln | -4x + C_1 | = t + C_2
\]
\[ -4x + C_1 = \pm e^{-4t} - 4C_2 \]

So,

\[ x = C_3 e^{-4t} + \frac{1}{4} C_1 \]

and

\[ y = -3x + C_1 = -3C_3 e^{-4t} + \frac{1}{4} C_1 \]

When you put in the initial conditions you find \( C_1 = 2, C_3 = 0 \).

Remember that you need to add \(+C\) with a new \( C \) every time you integrate.

0.5 Find all functions \( f \) from integers to real numbers so that

\[ f(n) = \frac{1}{2} f(n+1) + \frac{1}{2} f(n-1) - 1 \]

[Show first that \( f(n) = n^2 \) is a particular solution.]

To solve the homogeneous equation, try \( f = a^n \). If \( a \) is a double root then the second solution is \( f(n) = na^n \). The homogenous equation gives

\[ a^2 - 2a + 1 = 0 \]

This has only one root: \( a = 1 \). So, the solutions are \( f(n) = 1 \) and \( f(n) = n \). Thus the general solution is

\[ n^2 + bn + c \]

where \( b \) and \( c \) are constants. There is no constant in front of the particular solution.

0.6 (a) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) so that

\[ f''(x) + f'(x) + f(x) = 0 \]

Here you try \( f(x) = e^{\lambda x} \) and you find that

\[ \lambda^2 + \lambda + 1 = 0 \]

or

\[ \lambda = \frac{-1 \pm i\sqrt{3}}{2} \]

\[ e^{\lambda x} = e^{-x/2}(\cos \frac{\sqrt{3}x}{2} \pm i \sin \frac{\sqrt{3}x}{2}) \]

To get a real solution students correctly took a linear combination of the real and imaginary parts:

\[ f(x) = ae^{-x/2} \cos \frac{\sqrt{3}x}{2} + be^{-x/2} \sin \frac{\sqrt{3}x}{2} \]

(b) Find all functions \( f : \mathbb{Z} \to \mathbb{R} \) so that

\[ f(n+2) + f(n+1) + f(n) = 0 \]
You try $f(n) = a^n$ and you get
\[ a^2 + a + 1 = 0 \]
Or
\[ a = \frac{-1 \pm i\sqrt{3}}{2} \]
So, an arbitrary complex solution is given by
\[ f(n) = a \left( \frac{-1 + i\sqrt{3}}{2} \right)^n + b \left( \frac{-1 - i\sqrt{3}}{2} \right)^n \]
In order for this to be a real number it must be equal to its complex conjugate. So, $b = \bar{a}$. I.e., $a = c + id, b = c - id$ where
\[ c = f(0)/2 \]
\[ d = -f(1)\sqrt{3}/3 \]