MATH 56A: FALL 2006
HOMEWORK AND ANSWERS

2. Math 56a: Homework 2

p. 35 #1.1, 1.2, 1.3

1.1. Every morning a newspaper is added to a pile. With probability 1/3 the pile is emptied emptied. But if the pile has 5 newspapers, it is always emptied. Make this into a Markov chain.

This is a Markov process with states $0, 1, 2, 3, 4$ representing the number of newspapers in the pile in the evening. The transition matrix is

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1.2. Given that

$$P = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix}$$

What is the probability that $X_3 = 1$ given that $X_0 = 0$?

The answer is the $(0, 1)$ entry of P^3. Instead of doing this in the straightforward boring method I will use right eigenvectors: $Pv_i = \lambda_i v_i$.

The eigenvalues of P are $1, -5/12$ with right eigenvectors $v_0 = (1, 1)$ and $v_1 = (8, -9)$:

$$e_0 = \frac{9}{17}v_0 + \frac{1}{17}v_1$$

$$P^3 e_0 = \frac{9}{17}v_0 + \left(\frac{-5}{12}\right)^3 \frac{1}{17}v_1$$

whose 0th coordinate is

$$p_3(0, 1) = \frac{9}{17} + \left(\frac{-5}{12}\right)^3 \frac{8}{17} = \frac{109}{216}$$

1.3. Given that

$$P = \begin{pmatrix} .4 & .2 & .4 \\ .6 & 0 & .4 \\ .2 & .5 & .3 \end{pmatrix}$$

what is the long term probability of being in state 1?

The answer is

$$25/66 = 0.378787879$$

1) By directly raising this matrix to a high power.

By squaring the matrix 4 times you get

$$P^{16} = \begin{pmatrix} 0.378787879 & 0.257575758 & 0.363636364 \\ 0.378787879 & 0.257575758 & 0.363636364 \\ 0.378787879 & 0.257575758 & 0.363636364 \end{pmatrix}$$

So, the invariant probability distribution is

$$(0.378787879, 0.257575758, 0.363636364)$$

The first coordinate is the long term probability of being in state 1.

2) By directly computing the invariant distribution as a left eigenvector.

The invariant probability distribution is given by the equation: $(x, y, z)P = (x, y, z)$ with solution

$$\pi = (x, y, z) = \left(\frac{25}{66}, \frac{17}{66}, \frac{4}{11}\right)$$
$$= (0.378787879, 0.257575758, 0.363636364)$$