MATH 56A: FALL 2006
HOMEWORK AND ANSWERS

Math 56a: Homework 4

4. Homework 4 (Chap 2)

p. 59 #2.7, 8, 18

2.7. Are these positive recurrent, null recurrent or transient?

(a) This process is null recurrent:

$$p(x,0) = \frac{1}{x+2}, \quad p(x,x+1) = \frac{x+1}{x+2}$$

In this process, you keep coming back to state 0. However, to be recurrent you need to know that the probability of returning to 0 is 1. The probability that you will get to state n is

$$p(0,1)p(1,2)\cdots p(n-1,n) = \frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{n}{n+1} = \frac{1}{n+1}$$

Since this goes to 0, the probability of returning to 0 is 1. So, this is **recurrent**.

To see if it is positive recurrent you need to find an invariant distribution or show that one exists. This is a vector solution of the matrix equation

$$\pi P = \pi$$

so that the entries of π add to 1. But the matrix equation gives:

$$\pi(x+1) = \pi(x)p(x,x+1) = \frac{\pi(x)(x+1)}{x+2}$$

So, $\pi(x)(x+1) = C$ is constant. But this is impossible since

$$1 = \sum \pi(x) = \sum \frac{C}{x+1}$$

is a diverging sum. So, there is no invariant distribution. So, this process is **null recurrent**.

(b) This one is positive recurrent:

$$p(x,0) = \frac{x+1}{x+2}, \quad p(x,x+1) = \frac{1}{x+2}$$

This one has a higher probability of returning to 0 than the last one. So, it must also be recurrent. To see if is it positive recurrent we again look for an invariant distribution:

$$\pi(x+1) = \pi(x)p(x,x+1) = \frac{\pi(x)}{x+2}$$

$$\pi(x) = \frac{\pi(0)}{(x+1)!}$$

The equation $\sum \pi(x) = 1$ gives

$$\pi(0) = \left(\sum \frac{1}{(x+1)!}\right)^{-1} = (e-1)^{-1}$$

So,

$$\pi(x) = \frac{(e-1)^{-1}}{(x+1)!}$$

is an invariant distribution and the process is **positive recurrent**.

(c) This one is transient:

$$p(x,0) = \frac{1}{x^2+2}, \quad p(x,x+1) = \frac{x^2+1}{x^2+2}$$

Since the return to 0 probabilities converge the process is transient:

$$\sum_{x=1}^{\infty} \frac{1}{x^2+2} < \sum \frac{1}{x^2} = \frac{\pi^2}{6} < \infty$$

(Or use the integral test or the p-test for convergence.) To see that the process is transient, take a number n so that

$$\sum_{x=n}^{\infty} \frac{1}{x^2+2} < \epsilon$$

Then, once you reach state x, the probability that you will ever return to 0 is less than ϵ. Since there is only one communication class, you keep returning to x or higher and eventually you never return to 0.

2.8. Branching process.

(a) $p_0 = .25, p_1 = .4, p_2 = .35$

The extinction probability is the smallest positive solution of

$$a = \phi(a) = \sum p_i a^i = .25 + .4a + .35a^2$$

So,

$$a = \frac{.6 \pm \sqrt{.36 - .35}}{.7} = 1, \frac{5}{7}$$

The smaller number is the answer: $a = 5/7$.

(b) $p_0 = .5, p_1 = .1, p_3 = .4$

Here you get the cubic equation

$$.4a^3 - .9a + .5 = 0$$

But you can factor out $a-1$ since $a=1$ is always a solution. You get

$$4a^2+4a-5=0$$
$$a=\frac{\sqrt{6}-1}{2}\approx .7247$$

(c) $p_0=.91, p_1=.05, p_2=.01, p_3=.01, p_6=.01, p_{13}=.01$

Here the average number of offspring is

$$\sum ip_i=.29<1$$

Therefore, the probability of extinction is one.

(d) $p_i=(1-q)q^i$ for some $0<q<1$.

This time, the average number of offspring is

$$\mu=\sum ip_i=(1-q)\sum iq^i=\frac{q}{1-q}$$

This is ≤ 1 if $q\leq 1/2$. So $a=1$ in that case.

If $q > 1/2$ then the extinction probability is the solution of

$$a = \sum (1-q)q^i a^i = \frac{1-q}{1-qa}$$

which gives

$$a = \frac{1-q}{q}$$

2.18. This is a rigorous proof of Stirling's formula

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$$

(a)

$$\lim_{n\to\infty} \sum_{n \le k < n+a\sqrt{n}} p(n,k) = \int_0^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

The central limit theorem says:

Theorem 2.1. *If $Y_n = X_1 + X_2 + \cdots + X_n$ is the sum of i.i.d random variables with mean μ and standard deviation σ then the random variable*

$$\frac{Y_n - n\mu}{\sigma\sqrt{n}}$$

approaches a standard normal distribution in the sense that

$$\lim_{n\to\infty} \mathbb{P}(n\mu + b\sigma\sqrt{n} \le Y_n < n\mu + a\sigma\sqrt{n}) = \int_b^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

For the Poisson distribution we have $\mu = 1 = \sigma$. If we take $b = 0$ then the limit becomes:

$$\lim_{n\to\infty} \mathbb{P}(n \le Y_n < n + a\sqrt{n}) = \lim_{n\to\infty} \sum_{n \le k < n+a\sqrt{n}} p(n,k)$$

(b) We need to show that, for $n \le k < n + a\sqrt{n}$,

$$e^{-a^2} p(n,n) \le p(n,k) \le p(n,n)$$

Let $\delta = k - n$. Then

$$\frac{n^k}{k!} = \frac{n^n}{n!} \cdot \frac{n}{n+1} \cdot \frac{n}{n+2} \cdots \frac{n}{n+\delta} \ge \frac{n^n}{n!} \frac{n^\delta}{(n+\delta)^\delta}$$

But,

$$\frac{n^\delta}{(n+\delta)^\delta} = \frac{1}{(1+\delta/n)^\delta} \ge e^{-\delta^2/n}$$

since $1 + \delta/n \le e^{\delta/n}$ and

$$e^{-\delta^2/n} > e^{-a^2}$$

since $\delta^2 < a^2 n$. This shows that

$$p(n,k) = e^{-n}\frac{n^k}{k!} \geq e^{-n}\frac{n^n}{n!}e^{-a^2} \geq e^{-a^2}p(n,n).$$

The other inequality is easy.

(c) Finally we are supposed to conclude that

$$p(n,n) \sim \frac{1}{\sqrt{2\pi n}}$$

from which Stirling's formula follows.

From (a) and (b) we get

$$a\sqrt{n}e^{-a^2}p(n,n) - \epsilon \leq \int_0^a \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \leq a\sqrt{n}p(n,n) + \epsilon$$

where $\epsilon > 0$ is arbitrarily small. (This comes from replacing only the middle terms with its limit: If $a_n < b_n$ then $\lim a_n \leq \lim b_n$ but $\lim a_n \leq b_n + \epsilon$.)

Divide by a and take limit as $a \to 0$ gives

$$\sqrt{n}p(n,n) - \epsilon \leq \frac{1}{\sqrt{2\pi}} \leq \sqrt{n}p(n,n) + \epsilon$$

which is what we wanted to prove.