MATH 56A: FALL 2006
HOMEWORK AND ANSWERS

Math 56a: Homework 5

1. Homework 5 (Chap 3)

p. 84 #3.5, 8, 11, 12

3.5. Let X_t be a Markov chain with state space $S = \{1, 2\}$ and rates $\alpha(1, 2) = 1, \alpha(2, 1) = 4$. Find P_t.

The infinitesmal generator is

$$A = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$$

This matrix has eigenvalues $0, -5$ with corresponding right eigenvectors $(1, 1)^t, (1, -4)^t$ forming the matrix Q

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix}$$

$P_t = e^{tA} = Qe^{tD}Q^{-1} =$

$$\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-5t} \end{pmatrix} \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix}$$

So,

$$P_t = \frac{1}{5} \begin{pmatrix} 4 + e^{-5t} & 1 - e^{-5t} \\ 4 - 4e^{-5t} & 1 + 4e^{-5t} \end{pmatrix}$$

3.8. The infinitesmal generator is

$$A = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(a) Find the invariant distribution π.

This is the solution of $\pi A = 0$ normalized so that the sum of the coordinates is 1:

$$\pi = \frac{1}{38}(3, 7, 9, 19) \approx (.079, .184, .237, .5)$$

(b) If $X_0 = 1$ what is the expected amount of time until the first jump?

The change rate is 3 times per unit time. So, the expected wait is 1/3 of a unit of time.

(c) If $X_0 = 1$ what is the expected time until you reach state 4?

You take $\tilde{A} = A$ with the 4th row and 4th column deleted. Then you want to solve the equation

$$\tilde{A}b = -\bar{1}$$

or:

$$\begin{pmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{pmatrix} \begin{pmatrix} b(1) \\ b(2) \\ b(3) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The solution is easy: $b(1) = b(2) = b(3) = 1$. So, $b(x) =$ (expected time to get from x to 4) $= 1$ for $x = 1, 2, 3$.

3.11. X_t is the birth-death process with $\lambda_n = 1 + 1/(n+1)$ and $\mu_n = 1$. Is this positive recurrent, null recurrent or transient?

Since

$$\lambda_n = 1 + \frac{1}{n+1} = \frac{n+2}{n+1}$$

the product collapses:

$$\lambda_n \lambda_{n-1} \cdots \lambda_0 = \frac{n+2}{n+1} \cdot \frac{n+1}{n} \cdots \frac{2}{1} = n+2$$

The sum

$$\sum \frac{\lambda_n \lambda_{n-1} \cdots \lambda_0}{\mu_{n+1}\mu_n \cdots \mu_1} = \sum n+2 = \infty$$

So, the process is not positive recurrent. Also,

$$\sum \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1} = \sum \frac{2}{n+2} = \infty$$

So, the process is not transient. Thus, it must be **null recurrent**.

What about $\lambda_n = 1 - 1/(n+2)$?

This is almost the same thing:

$$\lambda_n = 1 - \frac{1}{n+2} = \frac{n+1}{n+2}$$

the product collapses again:

$$\lambda_n \lambda_{n-1} \cdots \lambda_0 = \frac{n+1}{n+2} \cdot \frac{n}{n+1} \cdots \frac{1}{2} = \frac{1}{n+2}$$

The sum

$$\sum \frac{\lambda_n \lambda_{n-1} \cdots \lambda_0}{\mu_{n+1}\mu_n \cdots \mu_1} = \sum \frac{1}{n+2} = \infty$$

So, the process is not positive recurrent. Also,

$$\sum \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1} = \sum \frac{n+2}{2} = \infty$$

So, the process is not transient. Thus, it must be **null recurrent**.

3.12. For $\lambda_n = n\lambda, \mu_n = n\mu$ what values of λ, μ make extinction probability 1?

In this problem we first have to make the Markov chain irreducible by changing λ_0 to be 1. Then the extinction probability is one if and only if the new irreducible chain is recurrent, i.e., not transient. So, we take the sum:

$$\sum \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1} = \sum \frac{\mu^n}{\lambda^n}$$

This converges (making the chain transient) if and only if $\mu < \lambda$. So, the extinction probability is one if and only if $\mu \geq \lambda$ and $\mu > 0$. (When $\mu > \lambda$ the chain is positive recurrent and the expected time to extinction is finite.)