6. Homework 6 (Chap 4)

p. 98 #4.1, 2, 3, 4

4.1. We have a simple random walk with absorbing walls on $\{0, 1, 2, \cdots, 10\}$ with payoff function:

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f(x):$	0	2	4	3	10	0	6	4	3	3	0

Find the optimal stopping time rule and the value function $v(x)$ which gives the expected payoff at each state.

This one is easy. You use the convex function rule. Interpolating linearly we get:

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f(x):$	0	2	4	3	10	0	6	4	3	3	0
$v(x):$	0	2.5	5	7.5	**10**	8.6	7.2	5.8	4.4	**3**	0

The optimal stopping rule is to stop at $x = 4, 9$ and continue otherwise (if you can).

4.3. (a) Add a cost function of $g(x) = .75$ at each move.

Now, when we have to subtract $g(x)$ at each single gap, $2g(x)$ at each double gap and $3g, 4g, 3g$ at each triple gap. If the gap were longer we would have to solve the linear recursion:

$$v(k+1) = \frac{1}{2}v(k) + \frac{1}{2}v(k+2) - g(k+1)$$

For constant g the solution, for a gap of $n-1$, is

$$v(k) = -k(n-k)g$$

(The particular solution is $v(k) = k^2 g$. The homogeneous solutions are $v(k) = 1, v(k) = k$.) This gives:

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f(x):$	0	2	4	3	10	0	6	4	3	3	0
$v_1(x):$	0	2.5	5	7.5	**10**	8.6	7.2	5.8	4.4	**3**	0
$v_2(x):$	0	**2**	4.666	7.333	**10**	8	**6**	4.5	**3**	**3**	0
$v_3(x):$	0	**2**	**4**	7	**10**	7.25	**6**	**4**	**3**	**3**	0
$v(x):$	0	**2**	**4**	6.25	**10**	7.25	**6**	**4**	**3**	**3**	0

The optimal stopping rule is to continue only at $x = 3, 5$.

(b) a discount rate of $\alpha = .95$.

By iteration we get the following:

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f(x):$	0	2	4	3	10	0	6	4	3	3	0
$v(x):$	0	**2**	4.14	6.71	**10**	7.6	**6**	4.55	3.58	**3**	0

The optimal stopping rule is to stop at $x = 1, 4, 6, 9$. You can get the exact value of the (present) value function $v(x)$ by solving the equation

$$v(x) = \max(f(x), \alpha(v(x-1) + v(x+1))/2)$$

to get

$$v(2) = \frac{1710}{413}, \quad v(3) = \frac{2774}{413}$$

and

$$v(7) = \frac{1881}{413}, \quad v(8) = \frac{1482}{413}$$

(c) with both cost $g(x) = .75$ and discount rate of $\alpha = .95$.

By iteration we get the following:

x :	0	1	2	3	4	5	6	7	8	9	10
$f(x)$:	0	2	4	3	10	0	6	4	3	3	0
$v(x)$:	0	**2**	**4**	5.9	**10**	6.85	**6**	**4**	**3**	**3**	0

The optimal stopping rule is to continue at $x = 3, 5$ and stop elsewhere. You can get the exact value of the (present) value function $v(x)$ by solving the equation

$$v(x) = \max(f(x), \alpha(v(x-1) + v(x+1))/2 - g(x))$$

So,

$$v(3) = .95(14/2) - .75 = 5.9$$

and

$$v(5) = .95(16/2) - .75 = 6.85$$

are exact.

4.2. Now you roll two dice and $f(x)$ is the sum of the two numbers except for $f(7) = 0$.

a) What is your expected payoff if you always stop after the first roll?

This is just

$$E = \sum p(x)f(x) = 210/36 = 35/6 \approx 5.8333$$

b) What is your optimal payoff?

The question itself is a hint. Instead of trying to compute $v_n(x)$ we compute the expected payoff E_n for v_n.

$$E_1 = \sum_{x\neq 7} p(x)12 = 10$$

Given E_n we can compute E_{n+1} by

$$E_{n+1} = \sum_{x\neq 7} p(x)\max(f(x), E_n)$$

which gives:

$$\begin{aligned} E_2 &= 8.4444 \\ E_3 &= 7.4691 \\ E_4 &= 7.001 \\ E_5 &= 6.806 \\ E_6 &= 6.7247 \end{aligned}$$

Once you realize that the optimal stopping time is to stop when you get more than 7 then you can calculate the expected value:

$$\mathbb{E}(f(X_T)) = 140/21 \approx 6.66667$$

Since this is more than 6, the strategy is correct.

4.4. a) Do it again with cost function $g = [2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1]$.

Start with $E_1 = 10$. And repeat:

$$E_{n+1} = \sum_{x\neq 7} p(x)\max(f(x), E_n - g(x))$$

This gives:

$$\begin{aligned} E_2 &= 7.5 \\ E_3 &= 6.3194 \\ E_4 &= 5.9977 \\ E_5 &= 5.9443 \\ E_6 &= 5.9398 \\ E_7 &= 5.9394 \\ E_8 &= 5.9394 \end{aligned}$$

So, $\mathbb{E} = 5.9394$. and the optimal strategy is to continue only if you get 2 or 3. You can solve for E by

$$E = (E - 2)(3/36) + 202/36$$

to get $E = 196/33 \approx 5.939394$ (If you get a 4 you should keep it instead of paying 2 and getting 5.9394 for a net of 3.9394)

b) with discount rate $\alpha = .8$

Start with $E_1 = 10$. And repeat:

$$E_{n+1} = \sum_{x \neq 7} p(x) \max(f(x), .8E_n)$$

This gives:

$$\begin{aligned} E_2 &= 7.2222 \\ E_3 &= 6.3272 \\ E_4 &= 6.1283 \\ E_5 &= 6.0949 \\ E_6 &= 6.0904 \\ E_7 = E_8 &= 6.0898 \end{aligned}$$

So, $\mathbb{E} = 6.0898$ and the optimal strategy is to continue only if you get 2,3 or 4. You can get the exact value for E by solving the equation

$$E = 6(.8E)/36 + 190/36$$

which gives

$$\mathbb{E} = 1900/312 \approx 6.0897$$

(If you get a 5 you should keep it instead of rolling again to get an expected discounted payoff of 80% of 6.0897 which would be 4.8718)

c) both.

Start with $E_1 = 10$. And repeat:

$$E_{n+1} = \sum_{x \neq 7} p(x) \max(f(x), .8E_n - g(x))$$

This gives:

$$\begin{aligned} E_2 &= 6.5278 \\ E_3 &= 5.8796 \\ E_4 &= 5.8529 \\ E_5 = E_6 &= 5.8523 \end{aligned}$$

So, $\mathbb{E} = 5.8523$ and the optimal strategy is to continue only if you get 2. You can get exact answers by solving for E:

$$E = (.8E - 2)/36 + 208/36$$

This gives

$$\mathbb{E} = 2060/352 \approx 5.8523$$

(With a 2 you should pay the fee of 2 and get a net payoff of

$$5.8523 \cdot 0.8 - 2 = 2.6818)$$

- What is your expected payoff if you always stop after the first roll?

In all three cases this is the same as before (since you don't pay to continue and you don't get discounted)

$$E = \sum p(x)f(x) = 210/36 = 35/6 \approx 5.8333$$