5. Homework 7 (Chap 5)

p. 125 #5.2,5,7,15

5.2. If X_t is a Poisson process with $\lambda = 1$ then find $\mathbb{E}(X_1 \mid X_2), \mathbb{E}(X_2 \mid X_1)$.

By the definition of a Poisson process, $\mathbb{E}(X_2 - X_1) = \mathbb{E}(X_1) = \lambda = 1$. Also, $X_2 - X_1$ is independent of X_1. So,

$$\mathbb{E}(X_2 \mid X_1) = \mathbb{E}(X_2 - X_1 \mid X_1) + \mathbb{E}(X_1 \mid X_1) = X_1 + 1$$

If the total $X_1 + (X_2 - X_1)$ is given then, by symmetry,

$$\mathbb{E}(X_1 \mid X_2) = \frac{1}{2} X_2.$$

5.5. Suppose that X_n is the number of individuals in the nth generation in a branching process. If the mean number of offspring is μ then show that

$$M_n = \mu^{-n} X_n$$

is a martingale wrt $X_0, X_1, \cdots$

By definition of μ we have

$$\mathbb{E}(X_{n+1} \mid X_n) = \mu X_n$$

So,

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = \mu^{-n-1} \mathbb{E}(X_{n+1} \mid X_n) = \mu^{-n-1} \mu X_n = M_n$$

and we see that M_n is a martingale.

5.7. Take the random walk on $\mathbb{Z}$ where the probability of going right at each step is $p < 1/2$ and the probability of going left is $1 - p$. Take $S_n = a + X_1 + \cdots + X_n$ where $0 < a < N$.

(a) Show that

$$M_n = \left[\frac{1-p}{p}\right]^{S_n}$$

is a martingale.

First note that

$$M_{n+1} = \left[\frac{1-p}{p}\right]^{S_n + X_{n+1}} = M_n \left[\frac{1-p}{p}\right]^{X_{n+1}}$$

And

$$\mathbb{E}\left(\left[\frac{1-p}{p}\right]^{X_{n+1}}\right) = \left[\frac{1-p}{p}\right] p + \left[\frac{1-p}{p}\right]^{-1} (1-p) = (1-p) + p = 1$$

So,

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n \mathbb{E}\left(\left[\frac{1-p}{p}\right]^{X_{n+1}}\right) = M_n$$

So, M_n is a martingale.

(b) Suppose that T is the first time that S_n reaches 0 or N. Compute $\mathbb{P}(S_T = 0)$.

By the OST we have

$$\mathbb{E}(M_T) = M_0 = \left[\frac{1-p}{p}\right]^a$$

But this expected value is also given by

$$\mathbb{E}(M_T) = \mathbb{P}(S_T = 0) + \left[\frac{1-p}{p}\right]^N (1 - \mathbb{P}(S_T = 0))$$

So,

$$\mathbb{P}(S_T = 0) = \frac{\left[\frac{1-p}{p}\right]^a - \left[\frac{1-p}{p}\right]^N}{1 - \left[\frac{1-p}{p}\right]^N} = \frac{(1-p)^N - (1-p)^a p^{N-a}}{(1-p)^N - p^N}$$

5.15. Suppose that M_n is a martingale. Suppose there exists $Y \geq 0$ so that $\mathbb{E}(Y) < \infty$ and $|M_n| < Y$ for all n. Then show that M_n are uniformly integrable.

Since $\mathbb{E}(Y) < \infty$, the size of the tail of Y goes to zero. I.e., for any $\epsilon > 0$,

$$\mathbb{E}(Y I_{Y>K}) < \epsilon$$

for K sufficiently large. But $|M_n| > K$ implies $Y > |M_n| > K$. So, the indicator function of $|M_n| > K$ is less than or equal to the indicator function for $Y > K$. So,

$$|M_n| I_{|M_n|>K} \leq Y I_{Y>K}$$

and

$$\mathbb{E}(|M_n| I_{|M_n|>K}) \leq \mathbb{E}(Y I_{Y>K}) < \epsilon$$

Since the same K works for all n we have uniform integrability.