

**MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK 2006**

Math 56a: Homework 8, Chap 6: Renewal

8. TIME IS MONEY (CHAP 6)

$M/G/1$ queueing is explained on p. 148.

8.1. The insurance company. We have an insurance company which starts with a certain amount of capital x . It has a constant income (premiums) which arrives at the rate of 1 (one unit money per unit time). For example, we can take units to be

$$1 \text{ day} = \$1,000,000$$

The insurance company settles (pays) claims from time to time. Assume the occurrence of claims is a Poisson process with rate λ . For example, it could be $\lambda = 1/10$ which means one claim every 10 days on average.

Let U_n be the amount of the n -th claim. (in millions of dollars). Assume that

$$\lambda \mathbb{E}(U_n) < 1$$

This means the company is profitable in the long run. For example, suppose that every claim is exactly 5 (million \$). Then the company expects to keep about half its premiums.

8.2. Conversion to queueing. Here is an outline of how you convert this to an $M/G/1$ -queueing problem. Imagine that the company puts its money into a safe. Every time a claim is paid, money is taken out of the safe and there is a “hole” where the bundle of cash used to be. For each claim there is a new hole. As the money comes in, the holes are filled (at the rate of 1 in the order that they were made). So, if the first claim is 5 million \$, it will take 5 days to fill that hole.

Now imagine that the holes are people standing in line. When the n -th hole is being filled, the n -th person is being served. The time it takes to fill the hole is equal to the amount of the claim. This is U_n , the service time for the n -th person.

Your homework this week is to complete this analogy to answer the questions, or at least convert the questions into queueing questions.

8.3. Homework questions. Suppose that $\lambda = 1/10$ and $U_n = 5$ is constant.

- (1) Complete the analogy so that the relevant questions about the insurance company are converted into questions about the queue.
- (2) The queue has an equilibrium distribution because it is positive recurrent. What does this mean in terms of the company and can you find the particular equilibrium for the given rate of claims.
- (3) What is the *bankruptcy distribution function*

$$B(x) := \mathbb{P}(\text{company will go bankrupt if its initial capital is } x)$$

(What is the meaning of it in terms of the queue and can you calculate it?)

Some students asked if the premiums arrive at a particular time each day. No, we are doing a continuous process. The premiums come in at a continuous rate all the time. Claims come in at the (exponential) rate of $\lambda = 1/10$ and they are paid immediately with cash. The company is bankrupt when it cannot pay a claim immediately. (It cannot borrow money.)

I will do the problem this weekend and let you know if you need more information or more formulas.

First we need to interpret the random variable Y_n . In the queue, this is the number of people still in line when the n th person has been served. Serving people in line means fully paying for claims. So, Y_n is the number of claims left to be fully paid when the n th claim has been fully paid. We know exactly how long this takes. So, Y_n times 5 million dollars is the amount of money that we are in the red for $5n$ days after the first claim came in.

During the 5 days that it takes to fully pay for the $n + 1$ st claim, the number of new claims that will come in is given by the Poisson distribution. So,

$$\mathbb{P}(Y_{n+1} = Y_n + k - 1) = e^{-5\lambda} \frac{\lambda^k}{k!} = e^{-1/2} \frac{\lambda^k}{k!} = p(k)$$

$$\begin{aligned} p(0) &= 0.60653066 \\ p(1) &= 0.30326533 \\ p(2) &= 0.075816332 \\ p(3) &= 0.012636055 \\ p(4) &= 0.001579507 \\ p(5) &= 0.000157951 \\ p(6) &= 1.31626E - 05 \end{aligned}$$

This means the transition matrix P has entries

$$p(i, j) = \begin{cases} p(j - i + 1) & \text{if } j \geq i - 1 \geq 0 \\ 1 & \text{if } (i, j) = (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

The equilibrium distribution for Y_* is some infinite vector

$$\pi = (\pi(0), \pi(1), \pi(2), \dots)$$

so that $\pi P = \pi$ (and $\sum \pi(n) = 1$). This gives an infinite sequence of equations:

$$\begin{aligned} \pi(0) &= \pi(1)p(0) \\ \pi(1) &= \pi(0) + \pi(1)p(1) + \pi(2)p(0) \\ \pi(2) &= \pi(1)p(2) + \pi(2)p(1) + \pi(3)p(0) \\ \dots & \dots \\ \pi(k) &= \pi(1)p(k) + \pi(2)p(k-1) + \dots + \pi(k+1)p(k) \\ \dots & \dots \end{aligned}$$

Since we know that

$$\pi(0) = \frac{1}{\mathbb{E}(\tau) + 1} = \frac{1 - \lambda\mu}{2 - \lambda\mu} = \frac{1}{3}$$

we can solve these equations one by one:

$$\begin{aligned} \pi(1) &= \frac{\pi(0)}{p(0)} &= 0.549573757 \\ \pi(2) &= \frac{\pi(1) - \pi(0) - \pi(1)p(1)}{p(0)} &= 0.081733307 \\ \pi(3) &= \frac{\pi(2) - \pi(1)p(2) - \pi(2)p(1)}{p(0)} &= 0.025192069 \\ \pi(4) &= \frac{\pi(3) - \pi(1)p(3) - \pi(2)p(2) - \pi(3)p(1)}{p(0)} &= 0.007272549 \\ \pi(5) &= \frac{\pi(4) - \pi(1)p(4) - \pi(2)p(3) - \pi(3)p(2) - \pi(4)p(1)}{p(0)} &= 0.002071164 \\ \pi(6) &= \dots &= 0.000589322 \end{aligned}$$

The numbers $\pi(n)$ represent the long term proportion of the time that the number of unpaid claims will be n .

The Markov chain method of analyzing the bankruptcy probability is to compute the probability that Y_n will reach k before reaching 0:

$$P_k := \mathbb{P}(Y_n \text{ reaches } k \text{ before reaching } 0 \mid Y_0 = 1)$$

This is the $(1, 2)$ entry of the matrix $(I - Q)^{-1}S$

$$P_2 = (1 - Q)^{-1}S_{12} = (1 - p(1))^{-1}(1 - p(0) - p(1)) = 0.129467$$

$$P_3 = \left(I_2 - \begin{pmatrix} p(1) & p(2) \\ p(0) & p(1) \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 - p(0) - p(1) - p(2) \\ 1 - p(0) - p(1) \end{pmatrix}_{12} = 0.038373$$

$$P_4 = \left(I_3 - \begin{pmatrix} p(1) & p(2) & p(3) \\ p(0) & p(1) & p(2) \\ 0 & p(0) & p(1) \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 - p(0) - p(1) - p(2) - p(3) \\ 1 - p(0) - p(1) - p(2) \\ 1 - p(0) - p(1) \end{pmatrix}_{12} = 0.0110$$

$$P_5 = 0.0031$$

$$P_6 = 0.0009$$

Suppose that the company starts with 10 million dollars. Then it will go bankrupt in the first round of claims if it has its first claim after $5k$ days ($k \geq 0$ and before $5k + 5$ days and Y_n goes up to $k + 2$). (And this is only approximate since it measures only whole numbers of claims and does not take into account how far the company has gone to partially paying off the n th claim).

$$\mathbb{P}(\text{first claim comes in between } 5k \text{ and } 5k + 5 \text{ days}) = e^{-k/2} - e^{-(k+1)/2}$$

k	$e^{-k/2} - e^{-(k+1)/2}$	P_{k+3}	$\mathbb{P}(\text{bankruptcy})$
0	0.39346934	0.038373347	0.015098736
1	0.238651219	0.010956401	0.002614759
2	0.144749281	0.00311059	0.000450256
3	0.087794877	0.000884294	7.76364E - 05
4	0.053250285	0.000251641	1.33999E - 05

$$0.053576273 \quad 0.018254786$$

So, the probability of bankruptcy is approximately 2% according to this analysis.

Most of you just took the number $P_3 = 3.8373347\%$ as the probability of bankruptcy during the first round of claims. This turns out to be more accurate. The exact bankruptcy probability (starting at 10 million) being 5.3039403% as calculated on the next page. The sum of numbers in the third column 5.3576273% is slightly too big probably because it counts things twice. For example, the second number 0.010956401 should be multiplied by $1 - 0.038373347$ to get the probability that the company survives the first round and then dies on the second. With these corrections the third column adds up to 0.052944573 which is more accurate. I need to think about it more to figure out if this is exactly the correct formula.

The exact bankruptcy probability has a formula very similar to the formula for the equilibrium distribution. Suppose that $B(n)$ is the probability of bankruptcy for the company assuming that it starts with $5n$ million \$. Then $S(n) = 1 - B(n)$ is the *survival* probability (the probability that the company will never go bankrupt).

For, example, $S(0)$ is the probability that the company can make it starting with zero capital. Obviously, it needs to survive for 5 days with zero claims. This has probability $p(0) = e^{-1/2}$. Then it has 5 million dollars and its survival probability is $S(1)$. So,

$$S(0) = p(0)S(1)$$

If the company has 5 million then it can survive one claim but not two in the next 5 days. So, it has a $p(0) + p(1)$ probability of surviving for 5 days. After that it has either $S(2)$ or $S(1)$ probability of survival. So,

$$S(1) = p(0)S(2) + p(1)S(1)$$

And so on.

$$\begin{aligned} S(0) &= S(1)p(0) \\ S(1) &= S(1)p(1) + S(2)p(0) \\ S(2) &= S(1)p(2) + S(2)p(1) + S(3)p(0) \\ &\dots \dots \\ S(k) &= S(1)p(k) + S(2)p(k-1) + \dots + S(k+1)p(k) \\ &\dots \dots \end{aligned}$$

If we knew what was $S(0)$ we could calculate the rest:

$$\begin{aligned} S(1) &= 1.648721271 \times S(0) \\ S(2) &= 1.893921193 \times S(0) \\ S(3) &= 1.969497401 \times S(0) \\ S(4) &= 1.991315048 \times S(0) \\ S(5) &= 1.997528541 \times S(0) \\ S(6) &= 1.999296506 \times S(0) \\ S(7) &= 1.999799737 \times S(0) \end{aligned}$$

If the company starts with a lot of money then its survival probability is close to 1. So, $S(0) = 1/2$ and $S(n)$ and $B(n) = 1 - S(n)$ are given by:

$$\begin{array}{ll} S(0) = 0.5 & B(0) = 0.5 \\ S(1) = 0.824360635 & B(1) = 0.175639365 \\ S(2) = 0.946960597 & B(2) = 0.053039403 \\ S(3) = 0.9847487 & B(3) = 0.0152513 \\ S(4) = 0.995657524 & B(4) = 0.004342476 \\ S(5) = 0.99876427 & B(5) = 0.00123573 \\ S(6) = 0.999648253 & B(6) = 0.000351747 \\ S(7) = 0.999899868 & B(7) = 0.000100132 \\ S(8) = 0.999971496 & B(8) = 2.85044E - 05 \end{array}$$

So, for example, if the company starts with 10 million dollars its probability of bankruptcy is 5.3039403%. Notice that if it starts with nothing it has a 50-50 chance of survival.