

**MATH 56A: STOCHASTIC PROCESSES
ANSWERS TO HOMEWORK 2006**

Math 56a: Homework 9, Chap 7: Reversible

9. HOMEWORK 9 (CHAP 7)

p. 170 #7.1, 7.10

and rewrite the ALOHA protocol clearly as a Markov process. In other words, make the question clear. You don't have to go through the answer (why it is null recurrent).

7.1. Show that every irreducible, discrete time, two-state Markov chain is reversible with respect to its invariant probability.

This just follows from the definitions. There are two states 1, 2. Irreducible means the in the transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

we have $p, q > 0$. The invariant distribution is a distribution π so that $\pi P = \pi$:

$$(\pi(1), \pi(2)) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = (\pi(1)(1-p) + \pi(2)q, \pi(1)p + \pi(2)(1-q)) = \pi$$

This implies that $\pi(1)p = \pi(2)q$ or

$$\pi(1)p(1, 2) = p(1, 2)\pi(2)$$

This is the balance equation showing that the Markov chain is reversible.

7.10. Let $\alpha(x, y)$ be a symmetric rate function on the edges of \mathbb{Z}^d . Suppose there are real numbers $0 < c_1 < c_2 < \infty$ so that for all x, y with $\|x - y\| = 1$,

$$c_1 \leq \alpha(x, y) \leq c_2$$

(a) Show that X_t is recurrent if $d = 1$ or 2 .

This uses the Theorem stated on page 167 and proved in the next few pages. The theorem is that if α, β are symmetric transition rates on a graph and $\beta(x, y) \leq \alpha(x, y)$ for all vertices x, y and α is recurrent then so is β .

For the first part we use the fact proved in class that the integer lattice in \mathbb{Z}^1 and \mathbb{Z}^2 are recurrent. This implies recurrence for a constant rate c_2 and the theorem implies that $\alpha \leq c_2$ is also recurrent.

(b) Show that X_t is transient for $d \geq 3$.

Same thing. If α were recurrent then the constant rate $c_1 \leq \alpha$ would also be recurrent. But we know that for a constant rate, \mathbb{Z}^d is transient $d \geq 3$.