Math 56a: Homework 9, Chap 7: Reversible

9. Homework 9 (Chap 7)

p. 170 #7.1, 7.10

and rewrite the ALOHA protocol clearly as a Markov process. In other words, make the
question clear. You don’t have to go through the answer (why it is null recurrent).

7.1. Show that every irreducible, discrete time, two-state Markov chain is reversible with
respect to its invariant probability.

This just follows from the definitions. There are two states 1, 2. Irreducible means the in
the transition matrix
\[ P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \]
we have \( p, q > 0 \). The invariant distribution is a distribution \( \pi \) so that \( \pi P = \pi \):

\[ (\pi(1), \pi(2)) \begin{pmatrix} 1-p \\ q \\ p \\ 1-q \end{pmatrix} = (\pi(1)(1-p) + \pi(2)q, \pi(1)p + \pi(2)(1-q)) = \pi \]

This implies that \( \pi(1)p = \pi(2)q \) or

\[ \pi(1)p(1,2) = p(1,2)\pi(2) \]

This is the balance equation showing that the Markov chain is reversible.

7.10. Let \( \alpha(x, y) \) be a symmetric rate function on the edges of \( \mathbb{Z}^d \). Suppose there are real
numbers \( 0 < c_1 < c_2 < \infty \) so that for all \( x, y \) with \( ||x = y|| = 1 \),

\[ c_1 \leq \alpha(x, y) \leq c_2 \]

(a) Show that \( X_t \) is recurrent if \( d = 1 \) or 2.

This uses the Theorem stated on page 167 and proved in the next few pages. The theorem
is that if \( \alpha, \beta \) are symmetric transition rates on a graph and \( \beta(x, y) \leq \alpha(x, y) \) for all vertices
\( x, y \) and \( \alpha \) is recurrent then so is \( \beta \).

For the first part we use the fact proved in class that the integer lattice in \( \mathbb{Z}^1 \) and \( \mathbb{Z}^2 \)
are recurrent. This implies recurrence for a constant rate \( c_2 \) and the theorem implies that
\( \alpha \leq c_2 \) is also recurrent.

(b) Show that \( X_t \) is transient for \( d \geq 3 \).

Same thing. If \( \alpha \) were recurrent then the constant rate \( c_1 \leq \alpha \) would also be recurrent.
But we know that for a constant rate, \( \mathbb{Z}^d \) is transient \( d \geq 3 \).