10. HOMEWORK 10 (CHAP 8A)

Three problems:

a) **reflection principle** Take the example 1 on page 179 and redo it for arbitrary starting point and any variance. I.e., suppose that X_t is Brownian motion with variance σ^2 starting at $X_0 = a$. Calculate the probability:

$$\mathbb{P}(X_s = a \text{ for some } 1 < s < t\ )$$

This is just a calculation. The point is to see why the values of a, σ^2 don't matter. We want the probability that Brownian motion with variance σ^2, starting at a, will return to a sometime between time 1 and time $t > 1$.

$$\mathbb{P}(X_s = a \text{ for some } 1 < s < t \,|\, X_0 = a) =?$$

By symmetry and the reflection principle this is

$$4\mathbb{P}(X_1 > a \text{ and } X_t < a \,|\, X_0 = a)$$

$$= 4\mathbb{P}(X_1 - X_0 = b > 0 \text{ and } X_t - X_1 < -b \,|\, X_0 = a)$$

(At this point I already made a irrelevant.) The probability for fixed b (in the interval $(b, b + db]$) is

$$\phi_{\sigma^2}(b) db\, \Phi_{\sigma^2(t-1)}(-b)$$

where

$$\phi_{\sigma^2}(b)\, db = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-b^2/2\sigma^2}\, db = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \phi_1(x)\, dx$$

(with $x = b/\sigma$) and $\Phi_{\sigma^2(t-1)}$ is the cumulative distribution function:

$$\Phi_{\sigma^2(t-1)}(-b) = \int_{-\infty}^{-b} \phi_{\sigma^2(t-1)}(x)\, dx = \int_b^{\infty} \phi_{\sigma^2(t-1)}(x)\, dx = \int_{b/\sigma\sqrt{t-1}}^{\infty} \phi_1(y)\, dy$$

where we used the "convert to standard normal" rule. Substituting $b = x\sigma$ makes this

$$\Phi_{\sigma^2(t-1)}(-b) = \int_{x/\sqrt{t-1}}^{\infty} \phi_1(y)\, dy$$

So, the answer is given by:

$$\mathbb{P}(X_s = a \text{ for some } 1 < s < t \,|\, X_0 = a) = 4\int_0^{\infty} \int_{x/\sqrt{t-1}}^{\infty} \phi_1(x)\phi_1(y)\, dydx$$

Both a and σ^2 are gone. So, the answer is the same as before.

b) **fractal dimension** Suppose that X_t is standard Brownian motion. Let Y_t be the continuous function:

$$Y_t = \max_{0 \le s \le t} X_s$$

Show that

1. Y_t is monotonically increasing.
2. Y_t is differentiable almost everywhere (except on a set W of measure zero) with derivative zero.
3. Calculate the box dimension of W.

You can use the theorem

Theorem 10.0.1. *If the dimension of a set $A \subset \mathbb{R}^d$ is less than d then it has measure zero.*

Proof. For ϵ small, we can cover A with $C\epsilon^{-D}$ cubes of size ϵ. The measure (d-dimensional volume) of these cubes is ϵ^d. So,

$$\mu(A) \leq Ce^{-D}\epsilon^d = C\epsilon^{d-D}$$

If $D < d$ then $\lim_{\epsilon \to 0} \epsilon^{d-D} = 0$. So, $\mu(A) = 0$. $\square$

The first two steps are easy. For the box dimension the answer is that W looks exactly like Z and therefore has dimension 1/2. You could argue this intuitively or you could prove it rigorously.

First you need the distribution function of Y_t:

$$\begin{aligned}\mathbb{P}(Y_t > b) &= 2\mathbb{P}(X_t > b) \quad \text{by the reflection principle} \\ &= 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)\end{aligned}$$

So, the distribution function of Y_t is

$$F_{Y_t}(b) = \mathbb{P}(Y_t \leq b) = 2\Phi\left(\frac{b}{\sqrt{t}}\right) - 1$$

and the density function is

$$f_{Y_t}(b) = \frac{2}{\sqrt{t}}\phi\left(\frac{b}{\sqrt{t}}\right)$$

You also need to realize that $Y_1 = \max_{0 \leq s \leq 1} X_s$ has the same distribution as its time reversal,

$$Y_1^{op} = \max_{0 \leq u \leq 1}(X_u - X_1)$$

Now, we want to calculate $\mathbb{P}(W \cap [1,t] \neq \emptyset)$. This event happens if

$$\max_{1 \leq s \leq t}(X_s - X_1) = b$$

and

$$\max_{0 \leq u \leq 1}(X_u - X_1) \leq b$$

The probability that this happens for b in $(b, b+db]$ is

$$\begin{aligned}f_{Y_{t-1}}(b)\,db \cdot F_{Y_1}(b) &= \frac{2}{\sqrt{t-1}}\phi\left(\frac{b}{\sqrt{t-1}}\right)db\,(2\Phi(b) - 1) \\ &= \int_0^\infty \int_0^{x\sqrt{t-1}} 4\phi(x)dx\phi(y)dy \\ &= 4\int_0^\infty \int_0^{\tan^{-1}\sqrt{t-1}} \frac{1}{2\pi}e^{-r^2/2}rdrd\theta \\ &= \frac{2}{\pi}\tan^{-1}\sqrt{t-1} = 1 - \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{t-1}}\end{aligned}$$

which is exactly the same as

$$\mathbb{P}(Z \cap [1,t] \neq \emptyset)$$

Therefore, W and Z have the same box dimension.

c) **challenge question** Why does it make sense to say that the infinitesmal generator of Brownian motion is $\frac{1}{2}\partial^2/\partial x^2$?

Hint: In the discrete case (discrete state space, continuous time), $p_t(x,y) = (e^{tA})_{xy}$ and the xy coordinate of the infinitesimal generator is given by

$$A_{xy} = \frac{\partial}{\partial t} p(x,y)$$

If $f_t(x)$ is the distribution of states at time t then

$$\frac{\partial}{\partial t} f_t(y) = \sum_x f_t(x) A_{xy}$$

The hint more or less gives the answer. You just need to formulate it. Here is one way.

The infinitesimal generator of a Markov process can be defined to be the "space operator" A satisfying the equation

$$\frac{\partial}{\partial t} f_t(x) = (Af_t)(x)$$

where $f_t(x)$ is probability density function of the process at time t. (This is the same as the distribution of states if the number of particles is very large.) A *space operator* is any linear function

$$A : C^\infty(S) \to C^\infty(S)$$

where S is the state space. The point is that Af depends only on the value of $f_t(x)$ for the fixed time t and variable x (whereas $\frac{\partial}{\partial t} f_t$ depends only on the value of $f_s(x)$ for s close to t and for x fixed, making $\frac{\partial}{\partial t}$ a *time operator*.) If S is discrete (finite or countably infinite) then any function $f : S \to \mathbb{R}$ is C^∞.

So, the equation:

$$\frac{\partial}{\partial t} f_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f_t(x)$$

governing Brownian motion makes $A = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ the infinitesimal generator.