OUTLINE OF HIGHER IGUSA-KLEIN TORSION

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ABSTRACT. I will give an outline of the definition and basic properties of the version of higher Reidemeister torsion developed by John Klein and myself using parametrized Morse theory and polylogarithms. I also explain how this higher torsion is calculated first from the definition and from the axioms. I also explain at the end how these axioms might be extended to the case of nontrivial representations which factor through finite groups.

CONTENTS

These are notes for three lectures given at the American Institute of Mathematics in Palo Alto on Oct 19, 20 and 22, 2009 on the subject of higher Reidemeister torsion. Since Sebastian Goette, Bruce Williams and Berhard Badzioch had already described the Bismut-Lott analytic torsion [2] and Dwyer-Weiss-Williams higher torsion [4], my job was a little easier.

John Klein in his PhD thesis constructed higher Reidemeister torsion using Morse theory and Waldhausen A theory [14]. We did the first calculation of higher torsion on circle bundles in [12] using a dilogarithm formula for pictures which I developed [8]. In my book [9] I extended this using higher polylogarithms and used the properties of higher torsion to apply this calculation to other bundles [10]. The axioms play a central role in comparing different definitions of higher torsion in the nonequivariant case [11] and I report below on the problem session in which we discussed how one might try to extend these axioms to the general case.

The American Institute of Mathematics and their sponsor, the National Science Foundation, deserve a lot of credit for making this workshop on higher Reidemeister torsion productive and successful.

1. Input-output

Monday evening. In the first short lecture, I went over the basic properties of the higher torsion: What is the "input" and "output" for this higher torsion invariant?

Input Suppose that $(F, \partial_0 F)$ is a *smooth manifold pair*, by which I mean a compact smooth n-manifold F and a compact $n-1$ dimensional submanifold $\partial_0 F$ of ∂F. Suppose $(E, \partial_0 E) \to B$ is a bundle of compact smooth manifolds with subbundle $\partial_0 E$ so that the fiber is a manifold pair $(F, \partial_0 F)$. Suppose we have one of the following conditions on the fiberwise homology.

a) We have a unitary representation $\rho : \pi_1 E \to U(m)$ with associated flat $\mathbb{C}^m$ bundle $\mathcal{F} \to E$ so that $H_*(F_b, \partial_0 F_b; \mathcal{F})$ is a trivial bundle over B, i.e., the holonomy is trivial or
b) The rational homology $H_*(F_b, \partial_0 F_b; \mathbb{Q})$ is a unipotent $\pi_1 B$ module which means it has a filtration by $\pi_1 B$-submodules so that the quotients have trivial action of $\pi_1 B$.

Output In case (a) we get cohomology classes

$$\tau_k(E, \partial_0 E; \rho) \in H^{2k}(B; \mathbb{R})$$

In case (b) we get only the degree $4k$ classes:

$$\tau_{2k}(E, \partial_0 E) \in H^{4k}(B; \mathbb{R})$$

Properties

The basic properties are as follows.

(0) If $f : B' \to B$ is a smooth mapping, $f^*(E, \partial_0 E)$ is the pull back of the bundle pair $(E, \partial_0 E)$ then

$$\tau_k(f^*(E, \partial_0 E); \rho \circ f_*) = f^*(\tau_k(E, \partial_0 E))$$

(1) $\tau_k(E \times D^N, \partial_0 E \times D^N) = \tau_k(E, \partial_0 E)$

(2) Under very special conditions, such as when one of the bundles is a linear sphere or disk bundle we have the following formula for the fiber product of bundles:

$$\tau_k(E \times_B E', \partial_0) = \chi(F, \partial_0 F)\tau_k(E', \partial_0 E) + \tau_k(E, \partial_0 E)\chi(F'\partial_0 F')$$

(3) If $\partial F = \partial_0 F \coprod \partial_1 F$ and $\partial_1 F = \partial_0 F'$ then

$$\tau_k(E \cup E', \partial_0 E) = \tau_k(E, \partial_0 E) + \tau_k(E', \partial_0 E' = \partial_1 E)$$

Theorem 1.1. *If $\Sigma \to E \to B$ is an oriented unipotent surface bundle then*

$$\tau_{2k}(E) = \frac{1}{2}(-1)^k \zeta(2k+1)\frac{\kappa_{2k}(E)}{(2k)!} \in H^{4k}(B; \mathbb{R})$$

2. Definition of higher torsion

Tuesday morning. On the second day I gave an outline of the definition and calculation of the I-Klein torsion. First, I gave an outline of the outline:

$$\begin{matrix} E \to B \\ \rho : \pi_1 E \to U(m) \end{matrix} \xrightarrow{\text{Morse theory}} \mathcal{W}h^h(\mathcal{M}_m(\mathbb{C}), U(m)) \xrightarrow{\text{Kamber-Tondeur}}$$

$$\tau^{IK}(E; \rho) \in H^{2k}(B; \mathbb{R}) \xrightarrow{S^1\text{-bundles}} \text{calculation of } \tau^{IK}$$

Here is an outline of the steps needed to define the higher torsion.

(1) Stabilize the bundle: $(E \times D^N, \partial_0 E \times D^N)$.
(2) Get a fiberwise framed function.
(3) Use (a) unitary rep $\rho : \pi_1 E \to U(m)$ with trivial action of $\pi_1 B$ on $H_*(F_b, \partial_0 F_b; \rho)$.
(4) By Morse theory we get a family $C_b, b \in B$ of acyclic based (modulo $U(m)$) chain complexes over $\mathbb{C}$. We obtain:

$$B \to \mathcal{W}h^h(\mathcal{M}_m(\mathbb{C}), U(m))$$

(5) Use 2-index theorem to get a family of invertible matrices g_b. These are well-defined up to left and right multiplication by unitary matrices (from the representation ρ). Smoothly interpolate to make g_b a smooth function of $b \in B$.
(6) Take $h_b = g_b g_b^* \Rightarrow h^* = h$. So, h^t is defined for $0 \le t \le 1$. h_b is well-defined up to conjugation by a unitary matrix.

(7) The higher torsion is given by the $2k$-form on B given by

$$\frac{1}{(2k+1)!2i^k}\int_0^1 Tr\left((h_b^{-t}dh_b^t)^{2k+1}\right) + \text{correction terms}$$

(8) The correction terms are polynomials in the real and complex parts of the entries of h_b defined recursivelly using cyclic homology. We get:

$$\tau_k^{IK} \in H^{2k}(B;\mathbb{R}).$$

2.1. **Framed function theorem.** Suppose that $p : (E, \partial_0 E) \to B$ is a smooth manifold bundle with compact fiber pair $(F, \partial_0 F)$. We would like to have a *fiberwise Morse function* by which we mean a smooth function $f : (E, \partial_0 E) \to (I, 0)$ which restricts to a Morse function

$$f_b : (F_b, \partial_0 F_b) \to (I, 0)$$

for each $b \in B$. However, such a mapping may not exist and may not be unique. We get around this problem using the Framed Function Theorem and stabilization.

Theorem 2.1 (Framed Function Theorem [6])**.** *Suppose that* $\dim F \geq \dim B$. *Then there exists a fiberwise framed function* $f : (E, \partial_0 E) \to (I, 0)$. *If* $\dim F > \dim B$ *then this fiberwise framed function is unique up to framed homotopy.*

If $\dim F < \dim B$ then we just replace E with $E \times D^N$ for some large N and use the stability of the higher torsion invariant.

Definition 2.2. A *framed function* on a smooth manifold M is a pair (f, ξ) where

(1) $f : M \to \mathbb{R}$ is a smooth function with only nondegenerate and birth-death critical points. Near a birth-death, by definition, f is given by

$$x_1^3 - \sum_{j=2}^{i} x_j^2 + \sum_{k=i+1}^{n} x_k^2 + C$$

(2) $\xi = (\xi_1, \xi_2, \cdots, \xi_i)$ is a framing (vector space basis) for the (-) eigenspace of $D^2 f$ at each critical point.

The topology on the space of framed functions is given by letting each framing vector ξ_j vary continuously and adding a new framing vector in the positive cubic direction at birth-death points. In the example above, we need $\xi_i = \frac{\partial}{\partial x_1}$.

2.2. **Fiberwise Morse theory.** Ignoring birth-death points, for each $b \in B$ we choose paths from the base point to each critical point. Then we get a free chain complex over $\mathbb{Z}[\pi_1 E]$. This gives a mapping:

$$B \to \mathcal{W}h(\mathbb{Z}[\pi_1 E], \pi_1 E)$$

Tensor with the representation $\rho : \pi_1 E \to U(m)$ to get a chain complex over $\mathbb{C}$ with a basis in every degree. Call this chain complex $C(b)$. Then the trivial action of $\pi_1 B$ gives a compatible familty of homotopy equivalences of $C(b) \simeq C(b_0)$. Take the mapping cone of this homotopy equivalence to get a family of *acyclic chain complexes* $CC(b)$. This construction was also explained by Bernhard Badzioch right before my talk.

$$B \to \mathcal{W}h^h(\mathcal{M}_m(\mathbb{C}), U(m))$$

We can replace this family of acyclic chain complexes by a family of invertible matrices by the 2-index theorem:

Theorem 2.3 (2-index theorem [9], [7])**.**

$$\mathcal{W}h^h(R, G) \simeq \mathcal{W}h^{[i,i+1]h}(R, G)$$

In general $\mathcal{W}h^h(R, G)$ is the space of acyclic based free chain complexes over the ring R stabilized by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix}$$

where $g \in G$ and G is a subgroup of the group of units of R. And with G-change of basis morphisms.

In more details, $\mathcal{W}h(R, G)$ is a simplicial category. (See [10] for full details.)

Objects in degree 0 are pairs (P, a_0) where $P = \coprod P_i$ is a finite graded poset (P is the set of Morse critical points of $f|F_b$ ordered by critical value and graded by index) and $a_0 \in \mathrm{End}(RP)$ is the boundary map $RP_i \to RP_{i-1}$ making a free chain complex (RP, a_0) over R with basis P_i in degree i.

Morphisms are given by

(1) G change of bases for RP_i. (Conjugation of a_0 by diagonal matrices with coefficients in G)
(2) G stabilization

$$(RP, a_0) \mapsto (PR, a_0) \oplus (\cdots \to R \xrightarrow{g} R \to \cdots)$$

(3) Order preserving bijections $P \to P'$.

Handle slides a_1 occur in objects of degree ≥ 1, chain homotopies a_2 occur in objects of degree ≥ 2, etc.

$\mathcal{W}h^h(R,G)$ is the simplicial full subcategory of acyclic objects.

We apply this to $R = \mathcal{M}_m(\mathbb{C})$, $G = U(m)$. Then the higher torsion is the pull-back of a universal class

$$\tau_k \in H^{2k}(\mathcal{W}h^h(\mathcal{M}_m(\mathbb{C}), U(m)); \mathbb{R})$$

along the mapping

$$B \to \mathcal{W}h^h(\mathcal{M}_m(\mathbb{C}), U(m))$$

given by the fiberwise framed function, the representation ρ and the canonical rational quasi-isomorphism $C(F_b) \simeq H_*(F_b)$.

The reason that this, in principle, defines a higher torsion invariant is because of the following theorem.

Theorem 2.4 (I-Klein [13], [9]). *For any associative ring R and subgroup G of the group of units of R, there is a homotopy fiber sequence*

$$\mathcal{W}h^h(R,G) \to Q(BG_+) \to \mathbb{Z} \times BGL(\infty, R)^+$$

2.3. **Dupont-Kamber-Tondeur form.** To get an explicit cohomology class we use the Dupont formulation [3] of the Kamber-Tondeur form which goes as follows. For each $b \in B$ we have an acyclic based chain complex which is nonzero in only two degrees:

$$\cdots \to \mathbb{C}^m \xrightarrow{g_b} \mathbb{C}^m \to 0$$

The boundary map which was called a_0 in some lectures is now just an invertible matrix g_b with complex entries which depends on $b \in B$. Let $h = gg^*$ where g^* is the *adjoint* or conjugate transpose of g. Then $h = h^*$ is positive definite. So h^t is well defined for any real number t. Now take the differential form

$$tr\left((h_b^{-t} dh_b^t)^{2k+1}\right)$$

and integrate over $0 \leq t \leq 1$. This gives a $2k$-form on B which is unfortunately not closed. However, if we add a correction term we get a closed form

$$\frac{1}{(2k+1)!2i^k}\int_0^1 Tr\left((h_b^{-t} dh_b^t)^{2k+1}\right) + \text{correction terms}$$

The correction terms are polynomials in the entries of $a_0 = g_b$ and the higher superconnection terms a_1, a_2 which are uniquely determined by g_t.

The incidence matrix g_t is only well defined up to left and right multiplication by locally constant unitary matrices $g_b \mapsto U' g_b U$ and by expansion $g_t \mapsto g_t \oplus U''$. However, these ambiguities disappear in the formula because:

$$h_b = g_b g_b^* \mapsto U' g_b \underline{UU^*}_{I_m} g_b^* U'^* = U' g_b g_b^* U'^*$$

So, $(h_b^{-t} dh_b^t)^{2k+1}$ becomes $U'(h_b^{-t} dh_b^t)^{2k+1} U'^*$ which has the same trace. Also when g_b becomes $g_b \oplus U$, $h_b = g_b g_b^*$ becomes $h_b \oplus I_1$. So, the differential form is well-defined.

3. Calculation of higher torsion

(1) Circle bundle formula
(2) Linear independence of polylogarithms over $\mathbb{C}(x)$.
(3) Computation of the Dupont-Kamber-Tondeur form.
(4) Framing Principle.

3.1. **Circle bundle formula.**

The fundamental calculation upon which all other calculations come from is the following.

Theorem 3.1. *Suppose that λ is a complex line bundle over B and $S^1(\lambda) \to B$ is the associated principal circle bundle. Let $S^1(\lambda)/Z_n$ be the associated S^1/Z_n bundle. This has a unitary representation ρ_ζ sending $s \in Z_n$ to ζ^s where $\zeta^n = 1$. Then*

$$\tau_k(S^1(\lambda)/Z_n; \rho_\zeta) = -\frac{n^k}{k!} L_{k+1}(\zeta) c_1(\lambda)^k$$

where $c_1(\lambda)$ is the first Chern class of λ and

$$L_{k+1}(\zeta) = \Re\left(\frac{1}{i^k}\sum_{n=1}^{\infty}\frac{\zeta^n}{n^{k+1}}\right)$$

is the polylogarithm function of ζ.

First we show, as a consequence of the transfer formula 3.2 that the higher torsion is a multiple of $n^k L_{k+1}(\zeta) c_1(\lambda)^k$ and then we calculate the coefficient. This is explained in my problem session report below. The linear independence lemma 3.3 below allows us to ignore the polynomial correction term.

Lemma 3.2 (Transfer formula)**.**

$$\tau_k(E/G; Ind_H^G V) = \tau_k(E/H; V)$$

if the finite group G acts freely and fiberwise on E and H is a subgroup of G.

Lemma 3.3. *The $kN+1$ functions of $x \in S^1$ given by 1 and $L_j(x\zeta)$ for $j = 1, 2, \cdots, k$ and $\zeta^N = 1$ are linearly independent over the function field $\mathbb{C}(x)$.*

3.2. **The calculation.** We take the kth exterior tensor power of λ over $B = (S^2)^k$ and calculate

$$\int Tr((h^{-t} dh^t)^{2k+1})$$

Imitating the method of Igusa-Klein (in the $k = 1$ case), we construct a fiberwise framed function on the circle bundle over $B = (S^2)^k$. We isolate the critical parameter values, take local coefficients in B. We need to smoothly interpolate between matrices of the form

$$g_t = \begin{pmatrix} 1 & 0 & -t_1 \\ -t_2 & 1 & 0 \\ 0 & -t_3 & 1 \end{pmatrix}$$

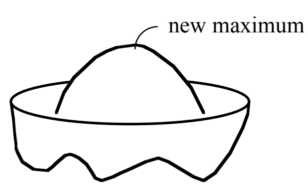
and we take $h_t = g_t g_t^*$ and take $Tr((h_t^{t_0} dh_t^{-t_0})^{2k+1})$ and integrate for $t \in I^{2k+1}$. This is the sum of a large number of $2k+1$-forms. But, after simplification, the integral of each form is the same up to sign! The polylogarithms arise because when we take the inverse of the matrix $g_t^{t_0}$ we get

$$\frac{1}{(1-\zeta)^{t_0}}$$

The final answer was given above.

3.3. **Framing Principle.** Finally, to get from circle bundles to higher dimensional bundles we use a very simple trick. Take the D^2 bundle associated to a complex line bundle λ over B. We have a fiberwise Morse function given by norm square. This is framed since it has only one critical point which is a minimum and it clearly has constant chain complex $C(b) = \mathbb{C}$ for every $b \in B$. So the torsion is trivial.

But we can also take a function which has local maxima along the boundary and at the center of each 2-disk and has minima along a circle just inside the boundary of the 2-disk (just like a hat with rim turned up and bumpy bottom part).



suspension of Morse function on a circle

The part where the hat touches your head is a circle bundle giving the torsion for the circle bundle with $\zeta = 1$ (trivial coefficients). This is

$$(-1)^{k+1}\frac{1}{(2k)!}\zeta(2k+1)c_1(\lambda)^{2k}$$

Since we know the torsion is zero, this bogus term must be cancelled by a contribution from the new maximum at the origin. The critical point at the origin is *not framed*. So, we get the *Framing Principle* which says that a non framed component of the Morse point set contributes minus the term above:

$$(-1)^k\zeta(2k+1)ch_{4k}(\lambda)$$

Theorem 3.4. *If $S^n \to E \to B$ is the sphere bundle associated to an oriented $n+1$ dimensional vector bundle ξ over B then*

$$\tau_{2k}(E) = (-1)^k\zeta(2k+1)\frac{ch_{4k}(\xi\otimes\mathbb{C})}{2}$$

4. Axioms for higher torsion

(1) Transfer
(2) The axioms
(3) Miller-Morita-Mumford classes
(4) Statement of theorem
(5) Extension to the relative case
(6) Fiberwise Morse functions
(7) Generalization to finite group actions

Consider *unipotent* smooth bundles:

$$X \to E \xrightarrow{p} B$$

with fiber X a *closed* oriented manifold with $\dim X = n$.

4.1. **Transfer.** $tr_B^E : H^*(E) \to H^*(B)$ is given by the composition

$$\begin{array}{ccc} H^*(E) & \xrightarrow{\quad tr_B^E \quad} & H^*(B) \\ & {\scriptstyle \cup\, e(TX)} \searrow \qquad \nearrow {\scriptstyle p_*} & \\ & H^{*+n}(E) & \end{array}$$

where $e(TX) \in H^n(E;\mathbb{R})$ is the Euler class of the vertical tangent bundle TX and $p_* : H^{*+n}(E) \to H^*(B)$ is given by integration along fibers. This satisfies the following properties.

(1) $tr_B^E = 0$ if n is odd.

(2) The composition $tr_B^E \circ p^* : H^*(B) \to H^*(B)$ is multiplication by $\chi(X)$.
(3) If D is a smooth bundle over E then

$$tr_B^D = tr_B^E \circ tr_E^D$$

4.2. **The axioms.**

Definition 4.1. For a fixed positive integer k we define a degree $4k$ *higher torsion invariant* to be any function τ which assigns to every unipotent smooth manifold bundle $E \to B$ with closed fibers X an element

$$\tau(E) \in H^{4k}(B;\mathbb{R})$$

which satisfies the following three conditions.

(0) (*naturality*) $\tau(f^*E) = f^*\tau(E) \in H^{4k}(B';\mathbb{R})$ if f^*E is the pull-back of E along a smooth mapping $f : B' \to B$.
(1) (*additivity*) Suppose that E_1, E_2 are two unipotent smooth manifold bundles over B so that the vertical boundaries of E_1, E_2 are fiberwise diffeomorphic. Let $E = E_1 \cup E_2$ be given by identifying these vertical boundaries. Then

$$\boxed{\tau(E) = \frac{1}{2}\tau(DE_1) + \frac{1}{2}\tau(DE_2)}$$

where DE_i is the fiberwise double of E_i.
(2) (*transfer*) Suppose that $S^m \to D \to E$ is an oriented linear m-sphere bundle and let $\tau_E(D) \in H^{4k}(E;\mathbb{R}), \tau_B(D) \in H^{4k}(B;\mathbb{R})$ be the higher torsion of D as a bundle over E and B respectively. Then

$$\boxed{\tau_B(D) = \chi(S^m)\tau(E) + tr_B^E(\tau_E(D))}$$

Note that $\chi(S^m) = 0$ if m is odd and $tr_B^E(\tau_E(D)) = 0$ if $n = \dim X$ is odd. This implies the following.

Proposition 4.2. *τ is a higher torsion invariant if and only if its odd and even parts τ^-, τ^+ are higher torsion invariants:*

$$\tau^-(E) = \begin{cases} 0 & \textit{if } \dim X \textit{ is even} \\ \tau(E) & \textit{if } \dim X \textit{ is odd} \end{cases}$$

$$\tau^+(E) = \begin{cases} \tau(E) & \textit{if } \dim X \textit{ is even} \\ 0 & \textit{if } \dim X \textit{ is odd} \end{cases}$$

$$\tau(E) = \tau^-(E) + \tau^+(E)$$

We call τ^- an *odd higher torsion invariant* and τ^+ an *even higher torsion invariant.*

4.3. Examples.

Theorem 4.3. *I-K torsion τ_{2k}^{IK} is a higher torsion invariant for every $k > 0$.*

And there is another invariant which satisfies these axioms.

Definition 4.4. For closed fibers X, the MMM classes are defined by

$$M_{2k}(E) = tr_B^E((2k)!ch_{4k}(TX))$$

Theorem 4.5. *$M_{2k}(E)$ is an even higher torsion invariant. In other words, it satisfies the axioms and is equal to zero when $\dim X$ is odd.*

Theorem 4.6 (Badzioch,Dorabiala,Klein,Williams [1])**.** DWW *smooth torsion satisfies the axioms. So, it is a scalar multiple of* I-K *torsion.*

This uses the uniqueness theorem for higher torsion invariants explained below and the fact that DWW torsion satisfies the property that

$$\tau^{DWW}(D) = \tau^{DWW}(E)$$

for any linear disk bundle D over E. This property characterizes τ^{IK} up to a scalar multiple and is obvious satisfied by τ^{DWW} since it is defined on the fiberwise normal disk bundle of E which is the same for D and E.

Theorem 4.7 (Goette)**.** *Analytic torsion is a linear combination of* IK *and* MMM *and therefore satisfies the axioms in unipotent case. So, nonequivariant unipotent analytic torsion is an odd torsion theory.*

See [5].

4.4. Statement of the theorem.

Theorem 4.8. [11] *For each $k > 0$ every higher torsion theory is a linear combination of the two examples given above:*

$$\tau = a\tau_{2k}^{IK} + bM_{2k}$$

equivalently:

1. *Every even torsion is a scalar multiple of M_{2k}*
2. *Every odd torsion is a scalar multiple of the odd part of τ_{2k}^{IK}.*

This implies that τ is completely determined by its value on two example, one with odd dimensional fiber and one with even dimensional fiber.

Let λ be the universal complex line bundle over $\mathbb{C}P^\infty$ and let $S^1(\lambda) \to \mathbb{C}P^\infty$, $S^2(\lambda) \to \mathbb{C}P^\infty$ be the associated S^1 and S^2 bundles. Since

$H^{2k}(\mathbb{C}P^\infty;\mathbb{R})$ is one-dimensional, all characteristic classes are proportional. So, for any higher torsion invariant τ, we have the following for some $s_1(\tau), s_2(\tau) \in \mathbb{R}$

$$\tau(S^i(\lambda)) = 2s_i ch_{4k}(\lambda)$$

(1) $s_2(M_{2k}) = (2k)!$
(2) $s_1(M_{2k}) = 0$ since M_{2k} is an even torsion invariant.
(3) $s_n(\tau_{2k}^{IK}) = \frac{1}{2}(-1)^{n+k}\zeta(2k+1)$. So, τ_{2k}^{IK} has an equal amount of even and odd torsion (with opposite sign).
(4) $s_2(\tau^{BG}) = 0$ since analytic torsion is an odd torsion invariant. [2]

4.5. **Extension to relative case.**

Theorem 4.9. *Any higher torsion invariant defined for closed fibers can be extended uniquely to a higher torsion invariant defined for unipotent bundle pairs*

$$(X, \partial_0 X) \to (E, \partial_0 E) \to B$$

satisfying relative versions of the axioms.

The relative invariant is related to the absolute invariant by:

$$\tau(E, \partial_0 E) = \tau(E) - \tau(\partial_0 E)$$

Here the fiber X of $E \to B$ may have a boundary giving a subbundle $\partial X \to \partial E \to B$. The bounded case is related to the closed manifold fiber case by the equation:

$$\tau(E) = \frac{1}{2}\tau(DE) + \frac{1}{2}\tau(\partial E).$$

The positive sign here is related to the Euler characteristic calculation $\chi(DE) = 2\chi(E) - \chi(\partial E)$.

4.6. **Fiberwise Morse functions.** Suppose that $(E, \partial_0 E) \to B$ has a fiberwise Morse function $(E, \partial_0 E) \to (I, 0)$ with m critical points $x_1, \cdots, x_m$ having distinct critical values $f_b(x_1) < f_b(x_2) < \cdots$. Then E has a fiberwise handlebody structure:

$$E = E_m \supset E_{m-1} \supset \cdots \supset E_0 = \partial_0 E$$

$$E_j = E_{j-1} \cup D(\xi_j) \times_B D(\eta_j)$$

where $E_{j-1} \cap D(\xi_j) \times_B D(\eta_j) = S(\xi_j) \times_B D(\eta_j)$. Here ξ_j is the linear bundle over B given by the negative eigenspace of $D^2 f_b$ at the j-th critical point x_j and η_j is the posive eigenspace bundle. The dimensions of these bundles are $i_j, n-i_j$ where i_j is the index of x_j and $n = \dim X$.

The additivity axiom in the relative case implies:

$$\tau(E,\partial_0)=\sum\tau(D(\xi_j)\times_B D(\eta_j),S(\xi_j)\times_B D(\eta_j)).$$

where the sum is over all critical points x_j. Each summand can be calculated by the formula

$$\begin{aligned}\tau(D(\xi)\times_B D(\eta),S(\xi)\times_B D(\eta)) &= (-1)^i\tau(D(\eta))+\tau(D(\xi),S^{i-1}(\xi))\\ &= (-1)^i(s_1+s_2)ch_{4k}(\eta)+(s_i-s_{i-1})ch_{4k}(\xi).\end{aligned}$$

where i is the index of the critical point and $s_i=s_1$ or s_2 depends on the parity of i.

Note that the parameters s_1,s_2 play very similar roles. In the lecture I pointed out that you can "convert even torsion into odd torsion" by the following trick.

Suppose that $\Sigma\to E\to B$ is an oriented surface smooth bundle. Let $D\to E$ be the fiberwise tangent circle bundle of E. Then $D\to B$ is a smooth bundle with 3-dimensional closed manifold fibers. Since $\chi(S^1)=0$ we have the transfer formula:

$$\tau_B(D)=tr_B^E(\tau_E(D))$$

and we know that

$$\tau_E(D)=2s_1ch_{4k}(T\Sigma)=2s_1\frac{e(T\Sigma)^{2k}}{(2k)!}$$

So,

$$\tau_B(D)=\frac{2s_1}{(2k)!}\kappa_{2k}$$

is proportional to the MMM class $M_{2k}(E)=\kappa_{2k}$.

4.7. **Generalization to finite group actions.** We want to generalize the axioms to the case of nontrivial representations. I proposed the following two axioms and in the problem session we explored decided that this was not sufficient and search for a third "continuity axiom." Also, we need to define the torsion on what might be called "unitary-potent" representations. John Klein suggested that we could assume that the base space B is always simply connected.

Suppose we are given a smooth bundle $E\to B$ with closed smooth manifold fiber X and a unitary representation $\rho:\pi_1E\to U(m)$ satisfying the condition that π_1B acts trivially on $H_*(X,\rho)$. Then we assume we have a characteristic class $\tau(E;\rho)\in H^{2k}(B;\mathbb{R})$ satisfying:

Naturality (with coefficients)

$$\tau(f^*E;f^*\rho)=f^*\tau(E;\rho)\in H^{2k}(B';\mathbb{R})$$

if f^*E is the pull-back of E along a map $f:B'\to B$.

We assume (geometric) additivity and transfer as before and we need (at least) two additional axioms:

Additivity (for coefficients) If $\rho = \bigoplus \rho_i$ then

$$\tau(E; \bigoplus \rho_i) = \sum \tau(E; \rho_i)$$

Transfer/induction If G is a finite group which acts freely and fiberwise on E, H is a subgroup of G and V is a unitary representation of H then torsion of the orbit bundles $E/G, E/H \to B$ are related by

$$\tau(E/G; Ind_H^G V) = \tau(E/H; V)$$

In the problem session we looked over Milnor's paper characterizing polylogarithms. He defines the *Kubert identity* for a function $f : (0,1) \to \mathbb{C}$ to be the equation

$$f(x) = m^{s-1} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)$$

for all positive integers m and all $x \in (0,1)$. If we change coordinates by letting $f(x) = L(e^{2\pi i x})$ then the Kubert identity becomes:

$$L(\xi^m) = m^{s-1} \sum_{\zeta^m=1} L(\xi\zeta)$$

Theorem 4.10 (Milnor [15])**.** *The set of continuous functions $f : (0,1) \to \mathbb{C}$ satisfying the Kubert identity is two dimensional and is spanned by the Bernoulli polynomial β_s and the polylogarithm $L_s(e^{2\pi i x})$ where*

$$L_s(\zeta) = \Re\left(\frac{1}{i^{s-1}} \sum_{n=1}^{\infty} \frac{\zeta^n}{n^s}\right)$$

The way that we reduce to a situation where we can apply this theorem is as follows. First, we restrict to the case of finite cyclic groups. By coefficient additivity we can restrict to one dimensional unitary representations $\rho_\zeta : \mathbb{Z}/n \to U(1)$ which is given by sending the generator of $\mathbb{Z}/n$ to ζ, an n-th root of unity. Then the axioms imply the following.

Lemma 4.11. *Suppose that $G = \mathbb{Z}/nm$ acts freely and fiberwise on $E \to B$ and $H = \mathbb{Z}/n$ and τ satisfies the axioms above. Then*

$$\tau(E/(\mathbb{Z}/n); \rho_{\xi^m}) = \sum_{\zeta^m=1} \tau(E/(\mathbb{Z}/nm); \rho_{\xi\zeta})$$

Proof. The representation ρ_ξ of $G = \mathbb{Z}/nm$ restricts to the representation ρ_{ξ^m} on $H = \mathbb{Z}/n$. Therefore, by Frobenius reciprocity we have

$$Ind_H^G \rho_{\xi^m} = \bigoplus_{\zeta^m=1} \rho_{\xi\zeta}$$

The formula above now follows from the new axioms. □

Next we have the following formula which holds in general for any characteristic class in degree $2k$.

Lemma 4.12. *Assuming only naturality, we have the following for any principal S^1 bundle $E \to B$ and any nth root of unity ξ.*

$$\tau(E/(\mathbb{Z}/nm); \rho_\xi) = m^k \tau(E/(\mathbb{Z}/n); \rho_\xi)$$

Proof. Let $E = S^1(\lambda)$ be the universal circle bundle over $B = \mathbb{C}P^\infty$. Then the bundle $E/(\mathbb{Z}/m)$ over $\mathbb{C}P^\infty$ is also a circle bundle. So, it is classified by a map

$$f_m : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

In degree 2 we can see (by looking at circle bundles over S^2) that this map is multiplication by m on H^2. Then it follows that f_m^* is multiplication by m^k on $H^{2k}(\mathbb{C}P^\infty; \mathbb{R})$. The classifying maps for $E/(\mathbb{Z}/nm)$ and $E/(\mathbb{Z}/n)$ are related by the equation

$$f_{nm} = f_n \circ f_m$$

The lemma follows. □

Putting the two lemmas together we get:

$$\tau(E/(\mathbb{Z}/n); \rho_{\xi^m}) = m^k \sum_{\zeta^m=1} \tau(E/(\mathbb{Z}/n); \rho_{\xi\zeta})$$

if m divides n. To eliminate the dependence on the number n, we should multiply both sides by n^k and change notation to:

$$\tau(E/\mathbb{Z}_\infty; \xi) = n^k \tau(E/(\mathbb{Z}/n); \rho_\xi)$$

whenever ξ is an nth root of unity. Then we get the equation

$$\tau(E/\mathbb{Z}_\infty; \xi^m) = m^k \sum_{\zeta^m=1} \tau(E/\mathbb{Z}_\infty; \xi\zeta)$$

In other words, $L(\xi) = \tau(E/\mathbb{Z}_\infty; \xi)$ satisfies the Kubert identity.

In the problem session we search for a general continuity axiom which would imply that the function $\tau(E/\mathbb{Z}_\infty; \xi)$ is continuous in ξ. Although the search was unsuccessful, we understand the problem better and can now formulate the question more precisely.

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