EXCEPTIONAL SEQUENCES, BRAID GROUPS AND CLUSTERS

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Abstract. Auslander Reiten quivers of type A_n and B_n can be viewed as braid diagrams and cluster mutations can be seen as homotopies of these braids. These are examples of more general theorems about the "dual braid monoid" and exceptional sequences proved by Bessis, Brady-Watt, Ingalls-Thomas and myself with Ralf Schiffler.

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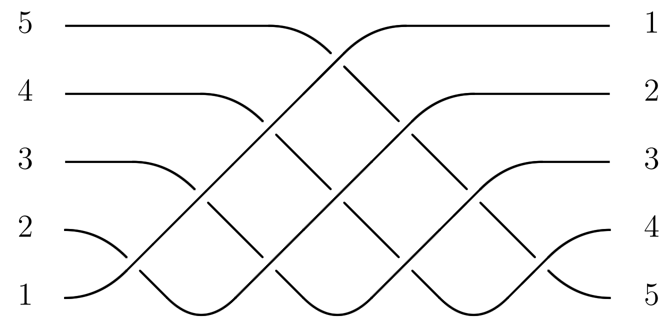
Introduction

These are notes for the talk I delivered at the XVIII Latin Ametican Algebra Colloquium held in August 2009 at Hotel Fonte Colina Verde, São Pedro, Brazil. This was a wonderful place with a great atmosphere. We all worked hard and had a very good time. Thank you for inviting us!

This lecture is about a topological interpretation of Auslander-Reiten quivers of type A_n and B_n and a topological recognition principle for cluster tilting objects of the cluster categories of these types as defined in [5]. First I showed the pictures and explained a little bit about the topology of braids and then I stated the theorems of Brady-Watt and Bessis and generalizations and tried to explain why one implies the other. These notes also contain an appendix giving a longer explanation about the "twist" in Auslander-Reiten translation which appears when we look at representations over nonalgebraically closed fields.

1. Topological AR-quivers

The following diagram shows what is known as the *Garside element* Δ in the braid group.

$\Delta =$ 

This can be viewed as a topological version of the Auslander-Reiten quiver of A_n where $n = 4$ in this case with the straight orientation:

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 4$$

I assume that vertices are always numbered so that arrows are decreasing. Thus $j \to i$ implies $i < j$.

This Garside element lies in the braid group $\mathcal{B}_5$ on $n+1 = 5$ strands. This is defined to be the fundamental group $\mathcal{B}_5 := \pi_1(C_5(\mathbb{C})/\Sigma_5)$ of the configuration space of $n+1 = 5$ distinct point in $\mathbb{C}$:

$$C_5(\mathbb{C}) = \mathbb{C}^5 \backslash \bigcup_{i<j} H_{z_i=z_j}$$

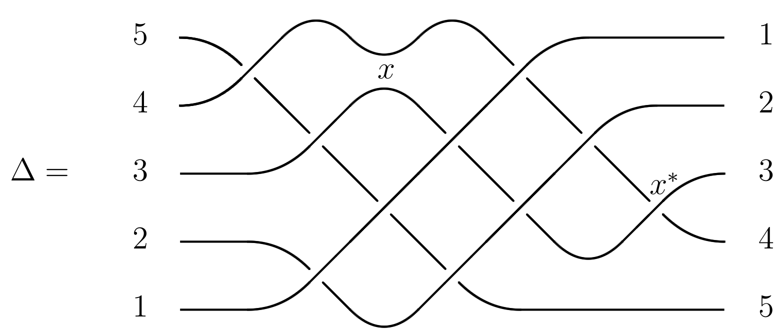
modulo the action of the symmetric group Σ_5.

The figure should be viewed as a family of vertical slices going from left to right. Each slice represents $\mathbb{C}$ with 5 distinct points on it. This is one point in the complement in $\mathbb{C}^5$ of the 10 hyperplanes corresponding to the 10 positive roots of the root system A_4.

If the orientation of the quiver is changed, say to $1 \leftarrow 4 \rightarrow 3 \rightarrow 2$ we get a braid which is homotopic to the braid above and therefore represents the same element Δ in the braid group. What I drew in the lecture was mutated further by removing the module/crossing at x and moving it to x^*.

To see that the two braids are equivalent it suffices to notice that

(1) They give the same permutation (also known as the longest word w_0 in the Weyl group).
(2) Strand 1 is on top, strand 2 is next and so on.



The two slides at the end of these notes illustrate all 14 cases of this for type A_3 and 10 out of the 20 cases of this in type B_3.

2. Theorem of Brady-Watt and Bessis

This braid interpretation of Auslander-Reiten quivers and cluster tilting objects is based on theorems of Brady-Watt [4] and Bessis [1]. But I should also mention that these in turn are based on the influential work of Birman, Ko and Lee[2] which does the case of A_n.

Let Q be a Dynkin diagram and let Φ_+ be the set of positive roots. For each $\beta \in \Phi_+$ we have the reflection $s_\beta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$s_\beta(x) = x - \frac{2(b,x)}{(\beta,\beta)}\beta$$

This is orthogonal reflection through the hyperplane

$$H_\beta = \{x \in \mathbb{R}^n \mid (\beta, x) = 0\}$$

Let $W = W(Q)$ be the *Weyl group* of Q. This is the subgroup of the orthogonal group generated by the set of reflections $S = \{s_\beta\}$. It is a finite group.

Definition 2.1. The *Artin braid group* $A(W)$ is defined to be the group with one generator σ_i for each vertex of Q and one relation for each pair of vertices i, j given as follows

$$A(W) = \left\langle \sigma_1, \cdots, \sigma_n \mid \underbrace{\sigma_i\sigma_j\sigma_i\cdots}_{m} = \underbrace{\sigma_j\sigma_i\sigma_j\cdots}_{m} \text{ for all } i -_m j \right\rangle$$

In particular σ_i, σ_j commute when $m = 2$. The number $m = m_{ij}$ is related to the multiplicity of the corresponding edge in the Dynkin diagram as follows.

Dynkin	m
$i \bullet \quad \bullet j$	2
$\bullet - \bullet$	3
$\bullet = \bullet$	4
$\bullet \equiv \bullet$	6

The usual braid group on $n + 1$ strands is equal to the Artin braid group of type A_n:

$$\mathcal{B}_{n+1} = A(A_n)$$

The corresponding Weyl group is the symmetric group $W(A_n) = \Sigma_{n+1}$. In this case σ_i is the basic generator given by twisting strands i and $i+1$ counterclockwise. (Some people like to define σ_i to go clockwise.)

Theorem 2.2 (Brady-Watt,Bessis)**.** *For any finite reflection group, the corresponding Artin braid group has the following presentation:*

$$A(W) = \left\langle \sigma_\beta, \beta \in \Phi_+ \mid \sigma_\beta\sigma_\gamma = \sigma_{s_\beta(\gamma)}\sigma_\beta \right\rangle$$

If we let $M(W)$ denote the group given by the presentation above, then this theorem says: $A(W) = M(W)$. Since the relations in $M(W)$ do not involve the inverses of the generators, this presentation defines a monoid $M_+(W)$ called the "Dual braid monoid" which explains the title of the paper [1]. This presentation has very good group theoretic properties as explained in the earlier work of Birman-Ko-Lee [2] in the case of the standard braid group.

In the rest of the lecture I tried to explain how this theorem, in the cases A_n and B_n, implies the topological interpretation of clusters given at the beginning and in what way Ralf Schiffler and I extended this to the general case in [8].

3. Exceptional sequences

We now go to the representation theory over a finite dimensional hereditary algebra. Since one of the basic examples that I am using is type B_n we cannot assume that the ground field K is algebraically closed. This lead to some unexpected discrepancies between the topology and algebra of Auslander-Reiten translation as I explained in my lecture.

3.1. **Basic definitions.** Suppose that H is a finite dimensional hereditary algebra over a field K.

Example 3.1. Take the modulated quiver

$$F_3 \xrightarrow{M_{32}} F_2 \xrightarrow{M_{21}} F_1$$

Here F_i are division algebras over K and M_{ji} is an $F_j - F_i$-bimodule. Thus

$$H = \begin{pmatrix} F_1 & 0 & 0 \\ M_{21} & F_2 & 0 \\ M_{32} \otimes M_{21} & M_{32} & F_3 \end{pmatrix}$$

A right H-module consists of F_i vector spaces V_i with F_i-linear maps

$$V_j \otimes_{F_j} M_{ji} \to V_i$$

Definition 3.2. A (finite dimensional right) H-module E is *exceptional* if

(1) $\mathrm{Ext}_H(E, E) = \mathrm{Ext}^1_H(E, E) = 0$
(2) $\mathrm{End}_H(E) = F_E$ is a division algebra.

In the first condition we drop the superscript ^{1} since H is hereditary making $\mathrm{Ext}^k_H = 0$ for $k \geq 2$. The second condition implies that E is indecomposable.

Definition 3.3. An *exceptional sequence* is $(E_1, E_2, \cdots, E_m)$ where

(1) E_i is exceptional for each i.
(2) $\mathrm{Hom}_H(E_j, E_i) = 0$ for $i < j$.
(3) $\mathrm{Ext}_H(E_j, E_i) = 0$ for $i < j$.

The exceptional sequence is called *complete* if $m = n$, the number of vertices of the quiver.

Example 3.4. I gave three examples of (complete) exceptional sequences assuming as always that arrows go from higher to lower vertices.

(1) $(P_1, P_2, \cdots, P_n)$, the projective modules
(2) $(S_n, S_{n-1}, \cdots, S_1)$, the simple modules in reverse order
(3) If $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ is a cluster in the cluster category $\mathcal{C}_H$ in admissible order (the order they appear in the Auslander-Reiten quiver in finite type or in the case that the modules are all preprojective), then

$$(|T_1|, \cdots, |T_n|)$$

is exceptional. Here $|T_i|$ denotes the unshifted modules: each T_i is a module $M[0]$ or a shifted projective $P[1]$. We define $|T_i|$ to be M or P respectively.

This absolute value function sending clusters to exceptional sequences is, in general, neither injective nor surjective.

3.2. **Braid action on exceptional sequences.** Crawley-Boevey [6] and Ringel [10] showed that the braid group $\mathcal{B}_n$ acts transitively on the set of complete exceptional sequences. In my lecture I defined the action explicitly following Bondal [3] and Gorodentsev [7].

Since $\mathcal{B}_n$ is generated by the basic braid generators σ_i we only need to define the action of this generator and its inverse. This is given as follows. I use the convention that the label does not change along

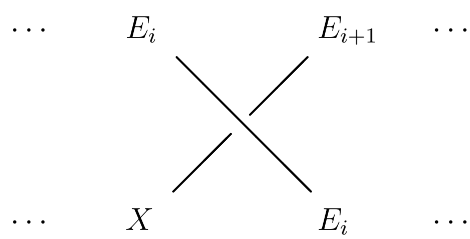


FIGURE 1. Action of σ_i on $(E_1, \cdots, E_n)$

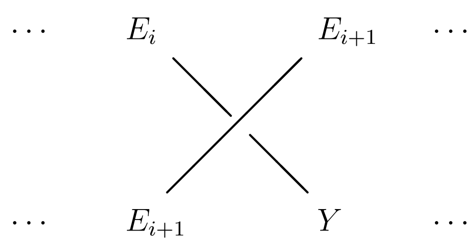


FIGURE 2. Action of σ_i^{-1} on $(E_1, \cdots, E_n)$

an unbroken line in the braid diagram. When the ith term E_i in an exceptional sequence "jumps over" the $i+1$st term, we replace E_{i+1} with a new module X which is uniquely determined up to isomorphism by the requirement that

$$(E_1, E_2, \cdots, E_{i-1}, X, E_i, E_{i+2}, \cdots, E_n)$$

be an exceptional sequence. Similarly, Y is uniquely determined by the requirement that

$$(E_1, E_2, \cdots, E_{i-1}, E_{i+1}, Y, E_{i+2}, \cdots, E_n)$$

be an exceptional sequence.

There are also formulas for X and Y which involves many cases. Suppose that $\mathrm{Hom}_H(E_i, E_{i+1}) \neq 0$ with $dim_{F_i}(\mathrm{Hom}_H(E_i, E_{i+1})) = m$. Then we have a homomorphism

$$f : \bigoplus_m E_i \to E_{i+1}$$

having the universal property that any homomorphism $E_i \to E_{i+1}$ factors uniquely through f.

Lemma 3.5. *This homomorphism is either a monomorphism or an epimorphism. We can let X be either the cokernel or kernel of this homomorphism respectively.*

Proof. Consider the exact sequence

$$0 \to X \to mE_i \xrightarrow{f} E_{i+1} \to Y \to 0$$

and let I be the image of f. Then $\mathrm{Ext}(E_{i+1}, I) = 0$ being a quotient of $\mathrm{Ext}(E_{i+1}, E_i) = 0$. So, we get another exact sequence

$$0 \to \mathrm{Hom}(E_{i+1}, I) \to \mathrm{End}(E_{i+1}) \to \mathrm{Hom}(E_{i+1}, Y) \to 0$$

Since $\mathrm{End}(E_{i+1})$ is a division algebra, either $\mathrm{Hom}(E_{i+1}, Y) = 0$ making $Y = 0$ and f an epimorphism or $\mathrm{Hom}(E_{i+1}, I) \cong \mathrm{Ext}(E_{i+1}, X) = 0$ which implies $\mathrm{Ext}(I, X) = 0$ which implies X is a direct summand of mE_i which implies $X = 0$ since f is universal. □

3.3. **Example: Auslander-Reiten translation.** Consider the case when the first $n-1$ terms in a (complete) exceptional sequence jump over the last term. In this case I claim that the new module Y is

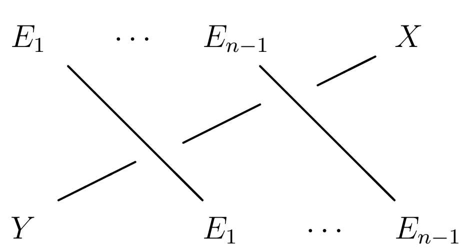


FIGURE 3. $Y \cong DTr\,X$

isomorphic to the Auslander-Reiten translate $\tau X = DTr\, X$ of X. The proof depends on the uniqueness of Y.

We are given that

$$\mathrm{Hom}_H(X, E_i) = 0 = \mathrm{Ext}_H(X, E_i)$$

By Auslander-Reiten duality these are isomorphic to

$$D\,\mathrm{Ext}_H(E_i, \tau X) = 0 = D\,\mathrm{Hom}_H(E_i, \tau X)$$

Since Y is uniquely determined up to isomorphism, we conclude that $Y \cong \tau X$.

I pointed out that in some sense, this isomorphism is not canonical since the action of the braid group on modules keeps track of the left action of the endomorphism ring which is always a division algebra since these modules are all exceptional. I used the example of A_2 with modulation to illustrate this. This is explained in detail in the Appendix.

4. Theorems about exceptional sequences

Theorem 4.1 (Crawley-Boevey, Ringel)**.** *The braid group acts transitively on the set of exceptional sequences.*

Definition 4.2. The *Coxeter element* $c \in W$ of the Weyl group is defined to be the product of the simple reflections:

$$c = s_1 s_2 \cdots s_n \in W$$

with inverse $c^{-1} = s_n \cdots s_1$. There is also a corresponding element in the braid group $A(W)$ and in $M(W)$:

$$\delta = \sigma_1 \sigma_2 \cdots \sigma_n$$

In the finite case (where we know that $A(W) \cong M(W)$ by [1],[4]) this is related to the Garside element by $\delta^h = \Delta^2$ where h is the order of the Coxeter element c.

Corollary 4.3. *If $(E_1, \cdots, E_n)$ form an exceptional sequence and β_i is the dimension vector of E_i then*

(1) $s_{\beta_1} \cdots s_{\beta_n} = c^{-1}$
(2) $\sigma_{\beta_n} \cdots \sigma_{\beta_1} = \delta$ *in* $M(W)$*.*

Proof. There is one generator in $M(W)$ for each exceptional module (and there are other generators from other modules). The relations in $M(W)$ exactly mirror the action of the braid group on exceptional sequences in reverse order and show that every elementary move (jumping one term over an adjacent term) gives the same product of generator in $M(W)$. Therefore the product is constant and equal to δ. □

Theorem 4.4. *The converse of the above theorem also holds. In other words both conditions are equivalent to the existence of an exceptional sequence of modules with dimension verctors* $\beta_1, \cdots, \beta_n$.

This was shown by Brady, Watt and Bessis in the finite case and extended to the affine case by Ingalls and Thomas [9] and by myself and Ralf Schiffler in the general case [8]. (We do not refer to the group $M(W)$ in our paper. We just show that Condition 1 above implies that there are corresponding modules forming an exceptional sequence. But Condition 2 easily implies Condition 1 and we already know from Crawley-Boevey and Ringel that an exceptional sequence satisfies Condition 2.)

The explanation of why this converse theorem implies the topological interpretation of clusters is the same as the explanation as to why Theorem 4.4 above is equivalent to Theorem 2.2 in the finite case. It depends on the following lemma and the fact that all modules are preprojective in the finite case.

Lemma 4.5. *Suppose that* $M = \tau^{-j} P_i$ *is a preprojective module and* $\dim M = \beta$. *(We write* $M = M_\beta$.*) Then*

(1) $s_\beta = w s_i w^{-1} \in W$ *where* $w = c^j s_1 \cdots s_{i-1}$.
(2) $\sigma_\beta = \gamma \sigma_i \gamma^{-1} \in M(W)$ *where* $\gamma = \delta^j \sigma_1 \cdots \sigma_{i-1}$.

Suppose that we have the topological version of the cluster category as shown in Figure 4. The circled point in the figure represents the

$\delta\Delta =$

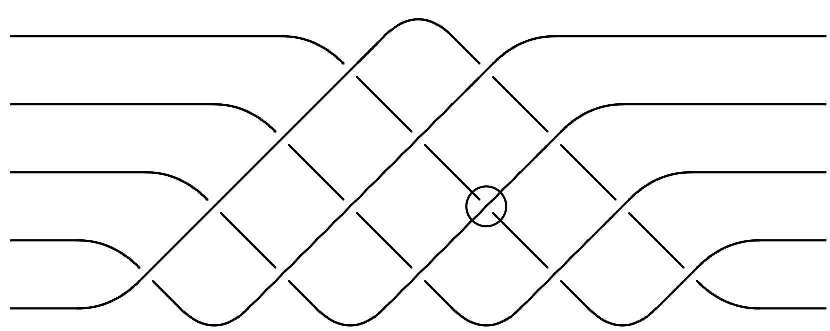


FIGURE 4. This is $\delta\Delta = \delta^2 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \in A(A_4) = M(A_4)$

module $M_\beta = \tau^{-2} P_2$. If we remove the corresponding letter, the first σ_2 in the word, then we get the following by Lemma 4.5 (2):

$$\delta\Delta = \sigma_\beta \delta^2 \sigma_1 \widehat{\sigma_2} \sigma_3 \sigma_1 \sigma_2 \sigma_1$$

If $(M_{\beta_1},\cdots,M_{\beta_n})$ form a cluster in admissible order then the corresponding letters can be deleted from the word $\delta\Delta$ starting from the right to get

$$\sigma_{\beta_n}\cdots\sigma_{\beta_1}(\delta\Delta \text{ with } n \text{ letters deleted})$$

We see that $\sigma_{\beta_n}\cdots\sigma_{\beta_1}=\delta$ if and only if ($\delta\Delta$ with n letters deleted) $=\Delta$. By Theorem 4.4 above, these conditions are both equivalent to the condition that $(M_{\beta_1},\cdots,M_{\beta_n})$ forms an exceptional sequence. However, if these are objects of the finite cluster category in admissible order, since homomorphism all go the right and extensions all go to the left, this is equivalent to the condition that $(M_{\beta_1},\cdots,M_{\beta_n})$ forms a cluster tilting object. This proves the geometric recognition principle for clusters stated at the beginning and illustrated in the figures at the end.

5. Appendix

This is about the "twisting" of the Auslander-Reiten functor $\tau = DTr$ in the nonalgebraically closed case. This arises from the fact that, for a bimodule over two division algebras, the left dual is not canonically isomorphic to the right dual.

5.1. **Example:** A_2**.** In my lecture I discussed the example of A_2 which is given by two finite dimensional division algebras F_1, F_2 over K and an F_2-F_1-bimodule M_{21} which is one dimensional on both sides:

$$A=\begin{pmatrix} F_1 & 0\\ M_{21} & F_2\end{pmatrix}$$

A right A-module consists of right F_i-vector spaces V_i which I wrote as $\binom{V_2}{V_1}$ and an F_1-linear map

$$V_2\otimes_{F_2} M_{21}\to V_1.$$

Any element $[\phi]\in M_{21}$ gives an isomorphism of division algebras

$$\phi: F_1\to F_2$$

by the equation $[\phi]x=\phi(x)[\phi]$. In the lecture I used the notation V_ϕ for $V\otimes_{F_2} M$ since the action of F_1 on V_ϕ is given by $v\cdot x = v\phi(x)$ and ${}_{\phi^{-1}}W = M\otimes_{F_1} W$ for left F_1-modules W.

The algebra $A=A_2$ has only three indecomposable modules: P_1, P_2, S_2 given by

$$P_1=\begin{pmatrix}0\\ F_1\end{pmatrix},\quad P_2=\begin{pmatrix}F_2\\ (F_2)_\phi\end{pmatrix}=\begin{pmatrix}F_2\\ {}_{\phi^{-1}}F_1\end{pmatrix},\quad S_2=\begin{pmatrix}F_2\\ 0\end{pmatrix}$$

These modules fit into the natural almost split sequence:

$$\begin{pmatrix} 0 \\ F_1 \end{pmatrix} \to \begin{pmatrix} {}_\phi F_2 \\ F_1 \end{pmatrix} \to \begin{pmatrix} {}_\phi F_2 \\ 0 \end{pmatrix}$$

or:

$$P_1 \to {}_\phi P_2 \to {}_\phi S_2$$

Next, I claimed that P_1 is not $DTr_\phi S_2$ at least not naturally. This is a simple calculation. First we compute $Tr({}_\phi S_2)$ (at vertex 1 only) by applying $\mathrm{Hom}_A(-, P_1)$ to this presentation:

$$\underbrace{\mathrm{Hom}_A(P_1, P_1)}_{=F_1} \leftarrow \underbrace{\mathrm{Hom}_A({}_\phi P_2, P_1)}_{=0}$$

This gives $Tr({}_\phi S_2) = \binom{0}{F_1}$ and

$$DTr({}_\phi S_2) = \begin{pmatrix} 0 \\ DF_1 \end{pmatrix} = DF_1 \otimes_{F_1} P_1.$$

5.2. **More general almost split sequences.** The existence of almost split sequences follows from Auslander-Reiten duality:

$$D\,\mathrm{Ext}^1_\Lambda(A, B) \cong \underline{\mathrm{Hom}}_\Lambda(\tau^{-1}B, A) \cong \overline{\mathrm{Hom}}_\Lambda(B, \tau A)$$

In the case that $A = \tau^{-1}B$ with endomorphism ring a division algebra $F_A = \mathrm{End}_\Lambda(A)$, this gives

$$\psi : \mathrm{Ext}^1_\Lambda(A, \tau A) = DF_A.$$

Thus there is a natural exact sequence of $F_A - \Lambda$-bimodules:

$$\boxed{\tau A \to E \to DF_A \otimes_{F_A} A}$$

corresponding to the adjoint of the inverse of ψ:

$$\widehat{\psi}^{-1} \in \mathrm{Ext}^1_\Lambda(DF_A \otimes_{F_A} A, \tau A) = \mathrm{Hom}_{F_A}(DF_A, \mathrm{Ext}^1_\Lambda(A, \tau A))$$

5.3. **Bimodule of injectives.** One other example that I stated in my lecture is the case of DTr of a projective in the bounded derived category.

Proposition 5.1. *In $\mathcal{D}^b(mod\, H)$, we have the natural isomorphism of H-bimodules:*

$$DTr \coprod P_i \cong \coprod DF_i \otimes_{F_i} I_i$$

Proof. The calculation of $DTrP_i$ is similar to the A_2 example. So, we will look at just the bimodule structure.

The projective module P_i has F_i at the top (at vertex i) and at vertex j it is the direct sum of all tensor paths:

$$(P_i)_j = \coprod M_{ii_1} \otimes M_{i_1 i_2} \otimes \cdots \otimes M_{i_{s-1} j}$$

Composition of paths gives the natural H-bimodule structure on $H = \coprod P_i$. The injective module I_j has F_j at bottom (at vertex j) and at vertex i the right dual of this $F_i - F_j$ bimodule:

$$(I_j)_i = \mathrm{Hom}_{F_j}((P_i)_j, F_j) = \coprod R_{ji_{s-1}} \otimes \cdots \otimes R_{i_1 i}$$

where $R_{pq} = \mathrm{Hom}_{F_p}(M_{qp}, F_p)$ is the right dual of M_{qp}. The right action of H is given by the natural contraction

$$R_{pq} \otimes_{F_q} M_{qp} \to F_p$$

given by $f \otimes x \mapsto f(x)$. However, we do not have a natural left action since the left dual is not canonically isomorphic to the right dual. But there is another natural contraction:

$$M_{qp} \otimes_{F_p} DF_p \otimes_{F_p} R_{pq} \to DF_q$$

given by $(x \otimes \phi \otimes f)(y) = \phi(f(yx))$ for $x, \phi, f, y \in M_{qp}, DF_p, R_{pq}, F_q$ respectively. This gives the left action of H on $\coprod DF_i \otimes I_i$. □

5.4. **Untwisting in separable case.** Using results from the lectures of Eduardo Tengan on central simple algebras we get the following.

Theorem 5.2. *Suppose that F is a finite dimensional division algebra over a field K whose center Z is a separable extension of K. Then there is a canonical isomorphism of F-bimodules $DF \cong F$.*

This theorem implies that, in the separable case, there is a canonical almost split sequence

$$\tau A \to E \to A$$

for any exceptional module A. In the purely inseparable case, I claimed in my lecture that this sequence is non-canonical.

Proof. Since F and DF are 1-dimensional on both sides, any isomorphism $F \cong DF$ is determined by the image of $1 \in F$ in DF. It is easy to see that this must be a nonzero K-linear function

$$tr : F \to K$$

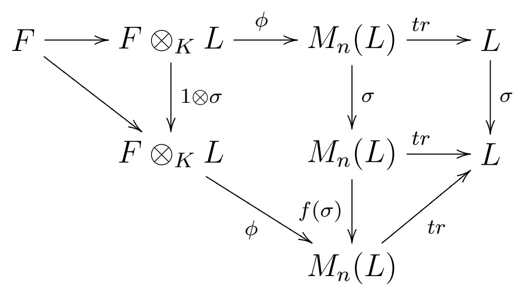
with the property that $tr(ab) = tr(ba)$. I call this a *trace map* and denote it by tr. If F is a finite separable field extension of K then the usual trace function $tr_K^F : F \to K$ has this property. Since the composition of trace maps is a trace map, it suffices to consider the case when K is the center of F. I.e., F is a central simple division algebra over K.

Suppose now that K is the center of F and L is a splitting field for F. Then I claim that the composition:

$$Tr : F \hookrightarrow F \otimes_K L \cong M_n(L) \xrightarrow{tr} L$$

is K-linear and has image K.

This map is nonzero and K-linear since F generates $F \otimes L$ and the trace $M_n(L) \to L$ given by the sum of diagonal entries is L-linear. In order to show that the image of Tr is equal to K it suffices to show that this image is invariant under the action of the Galois group $G = Gal(L/K)$, i.e. that $\sigma \circ Tr = Tr$ for all $\sigma \in G$. This follows from the following commuting diagram.



The commutativity of the lower right triangle follows from the fact that $f(\sigma)$ is an L-algebra automorphism of $M_n(L)$ which is an inner automorphism by weak Skolem-Noether (Lemma 5.2 in Tengan's lecture notes). The commutativity of the rest of the diagram was explained by Tengan. □

5.5. **Inseparable case.** Consider the category of finite field extensions (F, K) and isomorphisms of such pairs.

Theorem 5.3. *On this category there does not exist a natural isomorphism of F-bimodules $DF \cong F$.*

Proof. Let $F = \mathbb{F}_2(t, s)$ be the function field in two variables over the Galois field with two elements and let $K = \mathbb{F}_2(t, s^2)$. Let G be the group of $\mathbb{F}_2(t)$-algebra automorphisms of (F, K) given by $g(t) = t$ and

$$g(s) = t^k s + p(t)$$

where $k \in \mathbb{Z}$ and $p(t) \in \mathbb{F}_2(t)$. We can also write: $g = (k, p(t))$ and we see that G is a semidirect product $G \cong \mathbb{Z} \ltimes \mathbb{F}_2(t)$. Any natural isomorphism $F \cong DF$ would send $1 \in F$ to a nonzero K-linear mapping $F \to K$ which is G-equivariant. I claim this is impossible.

Suppose that $\gamma : F \to K$ were such a mapping. As a K-vector space F has basis $\{1, s\}$. So γ is determined by $\gamma(1), \gamma(s)$. If $g = (k, p(t)) \in G$ and $\gamma(1) = h(t, s^2)$ then

$$g\gamma g^{-1}(1) = g(h(t, s^2)) = h(t, g(s)^2) = \gamma(1)$$

which means that h does not involve s. So, $\gamma(1) = h(t)$. $g\gamma g^{-1}(s)$ can be calculated as follows. First, $g^{-1}(s) = t^{-k}s + t^{-k}p(t)$ and

$$\gamma g^{-1}(s) = t^{-k}f(t, s^2) + t^{-k}p(t)h(t).$$

This makes

$$\begin{aligned} g\gamma g^{-1}(s) &= t^{-k}f(t, g(s)^2) + t^{-k}p(t)h(t) \\ &= t^{-k}f(t, t^{2k}s^2 + p(t^2)) + t^{-k}p(t)h(t) = \gamma(s) \end{aligned}$$

Consider $\gamma(s) = f(t, s^2)$ as a rational function of s^2 with coefficients in $\mathbb{F}_2(t)$ and look at its degree d and leading coefficient $a(t)$. If $d > 0$ then the leading coefficient of $g\gamma g^{-1}(s)$ is $t^{k(2d-1)}a(t) \neq a(t)$. If $d = 0$ then the leading coefficient of $g\gamma g^{-1}(s)$ is $t^{-k}(a(t) + p(t)h(t))$ which cannot be equal to $a(t)$ for all k. If $d < 0$ then we must have $h(t) = 0$ to kill the constant term of $g\gamma g^{-1}(s)$. Then the leading coefficient is again $t^{-k}(a(t) + p(t)h(t))$ which cannot be equal to $a(t)$ for all k. Therefore, we have $f(t, s^2) = 0$ and $h(t) = 0$ making $\gamma = 0$ the only possibility. □

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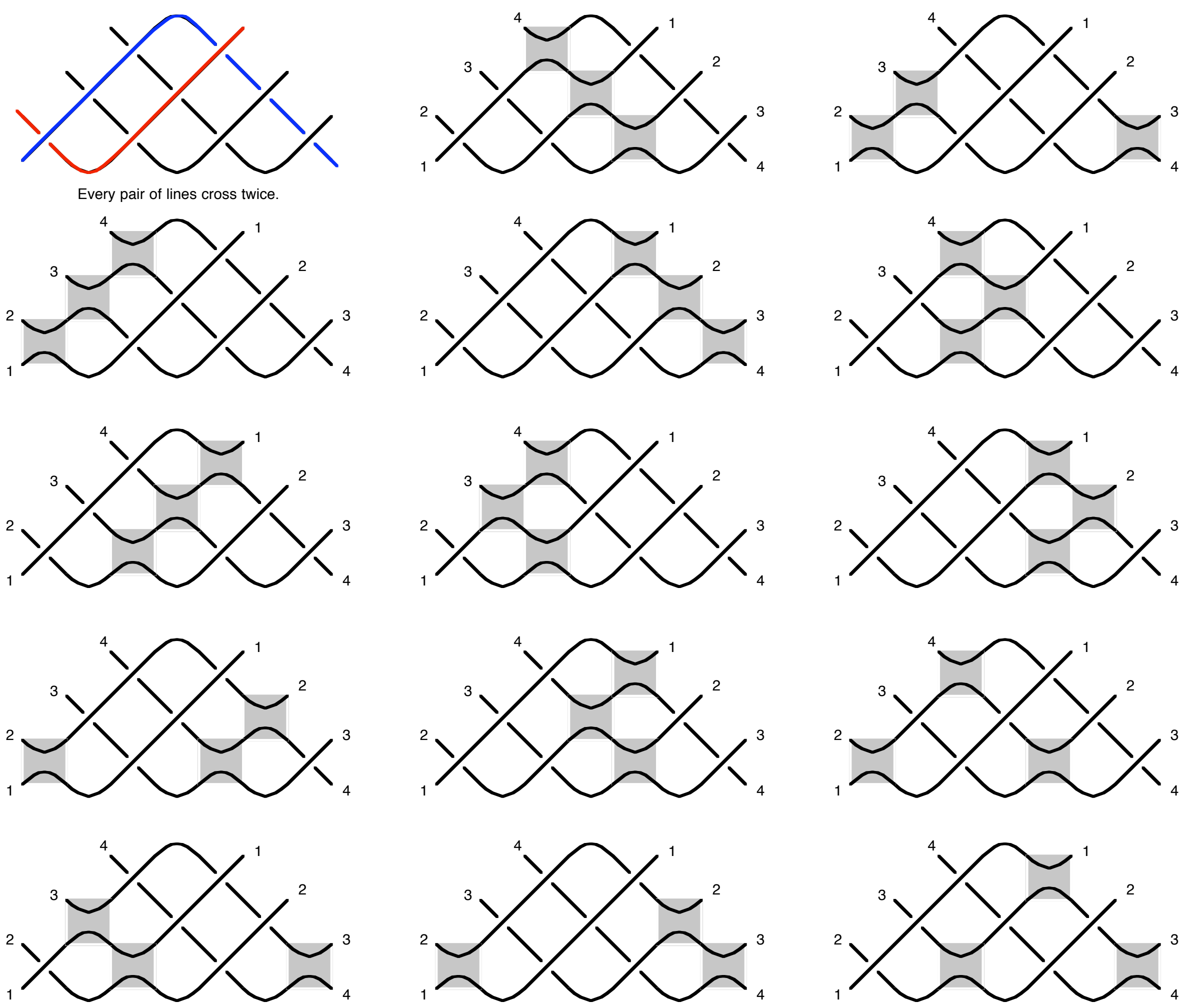
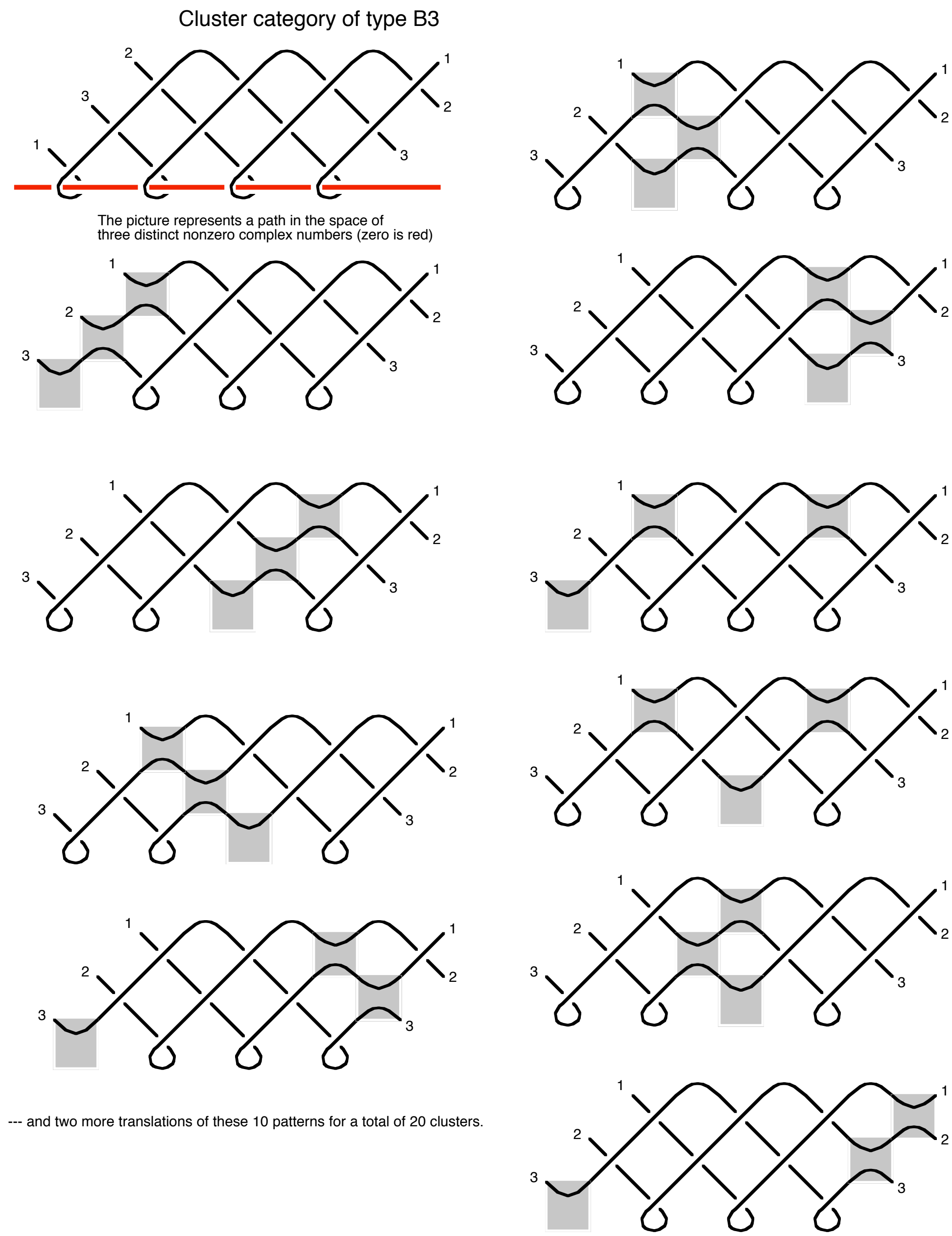
Cluster category of type A3



Theorem: The resulting braid is equal to the Garside element
if and only if
the removed crossings form a cluster.

These are all 14 clusters of type A3.


Cluster category of type B3
The picture represents a path in the space of
three distinct nonzero complex numbers (zero is red)
--- and two more translations of these 10 patterns for a total of 20 clusters.