

# THE CATEGORY OF NONCROSSING PARTITIONS

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ABSTRACT. In [12], picture groups are introduced and the cohomology of the picture group of type  $A_n$  with straight orientation is computed. In this paper, we give an elementary combinatorial interpretation of the category associated to  $A_n$  and prove that the classifying space of this category is a  $K(\pi, 1)$ . The objects of the category are the classical noncrossing partitions introduced in [18]. The morphisms are binary forests. This paper is independent of the later papers in this series except for the last section in which we compare our category with the one in [8].

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## INTRODUCTION

The concept of “pictures” was introduced by the author in his PhD thesis [9] to define the algebraic K-theory invariant for  $\pi_1$  of the space of pseudoisotopies of a smooth manifold. They are combinatorially equivalent to well-known objects in topology called “spherical diagrams” which are also called “diagrams of relations”. See [6], [21] for some of the earlier works on these diagrams. Later, in [10], pictures were used to prove the  $k$ -slice conjecture for Milnor’s  $\bar{\mu}$  link invariants. In [13], the sequel of [16], propictures are introduced. Canonical propictures associated to quivers of type  $\tilde{A}_n$  are constructed and shown to be related to periodic trees and cluster tilting objects of the quiver.

In [12], picture groups are introduced. These are the universal groups associated to the canonical semi-invariant picture of any modulated quiver of finite type. The special case of  $A_n$  with straight orientation is studied. An explicit model for the classifying space of the associated picture group  $G(A_n)$  is constructed by pasting together Stasheff associahedra. Using this model, which we call the “picture space”, the integral cohomology of  $G(A_n)$  is computed. It is shown to be free abelian in every degree with rank equal to the “ballot numbers”.

The paper [12] assumes that the picture space  $X(A_n)$  is a  $K(\pi, 1)$  for the picture group  $G(A_n)$ . The purpose of the present paper is to prove this basic fact. The proof is based on the following theorem of Gromov [7].

**Theorem 0.1** (Gromov). *A simply connected cubical space is  $CAT(0)$  if and only if the link of every vertex is a flag complex.*

Since  $CAT(0)$  spaces are contractible, we obtain the following conclusion.

**Corollary 0.2.** *If a connected cubical space has the property that the link of every vertex is a flag complex then it is a  $K(\pi, 1)$ .*

A flag complex is a simplicial complex with the property that every set of  $n + 1$  vertices which are pairwise connected by edges spans an  $n$ -simplex. It is clear that the link of every vertex of a cubical complex is a simplicial complex. But it is not always a flag complex.

Noncrossing partitions were introduced by Kreweras [18] and later generalized in [2], [1]. The relationship to representation theory originates in [17]. In [8] generalized noncrossing partitions become objects of a category. In this paper we used the classical noncrossing partitions of a single ordered set. This is a simplification of a special case of a construction from [14]. It is self-contained except for the last section.

The contents of this paper are as follows. In Section 1 we define noncrossing partitions on a finite totally ordered set  $V$  with  $n$  elements. These are the objects of the category  $\mathcal{NP}(n)$ . We call two parts of a partition *adjacent* if merging those parts produces another noncrossing partition. We consider trees having parts of a partition as vertices and edges connecting certain adjacent parts.

We define a morphism  $\mathcal{S} \rightarrow \mathcal{R}$  to be a binary forest consisting of a binary tree on each “parallel set” in  $\mathcal{S}$  relative to  $\mathcal{R}$ .

Section 2 is devoted to proving that composition of morphisms as given in Section 1 is well-defined and associative. In the language of Section 3, this depends on “backward links” of objects being flag complexes.

In Section 3 we define *cubical categories*. These describe general properties of the category  $\mathcal{NP}(n)$  which insure that the classifying space of the category is cubical. The local

properties of category, as proven in detail in Section 2, are shown to imply that this cubical space is locally  $CAT(0)$  and therefore a  $K(\pi, 1)$ .

In Section 4 we compute the fundamental group of the classifying space  $B\mathcal{NP}(n)$  of the category  $\mathcal{NP}(n)$  and show that it is equal to the picture group defined in [12].

In Section 5 we use cluster categories to compare the category of noncrossing partitions in this paper with the one given by Hubery and Krause in [8] which we will call  $HK$ . Basically the statement is that there is a category of cluster categories and a contravariant functor from that category to  $HK$ . The root system of the quiver  $A_{n-1}$  with straight orientation gives an object in  $HK$ . Our category  $\mathcal{NP}(n)$  is equivalent to a full subcategory of the comma category over this single object.

## 1. NONCROSSING PARTITIONS AND BINARY FORESTS

The category of noncrossing partitions will be defined in several steps. In this section we define the objects and morphisms of the category. In the next section we derive a formula for composition of morphisms.

The objects of our category will be noncrossing partitions of a finite totally ordered set. Its morphisms will be given by “binary forests.”

**1.1. Noncrossing partitions.** A *partition* of any set  $V$  is a set  $\mathcal{S}$  whose elements are disjoint subsets  $S_i \subseteq V$  with the property that  $V = \coprod S_i$ . The elements  $S_i \in \mathcal{S}$  are called the *parts* of the partition  $\mathcal{S}$ . Assuming that  $V$  is finite, the difference between the cardinality of  $V$  and that of  $\mathcal{S}$  will be called the *rank* of the partition.

$$rk \mathcal{S} := |V| - |\mathcal{S}|$$

For example, there is a unique partition of  $V$  of rank 0 given by partitioning  $V$  into singletons. We say that a partition  $\mathcal{R}$  is a *refinement* of  $\mathcal{S}$  if every part of  $\mathcal{R}$  lies in a part of  $\mathcal{S}$ . We say that  $\mathcal{S}$  is obtained from  $\mathcal{R}$  by *merging* parts of  $\mathcal{R}$  together. It is clear that, if  $\mathcal{R}$  is a refinement of  $\mathcal{S}$  and the difference between their ranks is  $k$  then  $\mathcal{R}$  can be obtained from  $\mathcal{S}$  in  $k$  steps where, in each step, two parts of the partition are merged.

When  $V$  is totally ordered, a partition of  $V$  is called *noncrossing* if there do not exist  $a < b < c < d$  so that  $a, c$  lie in one part and  $b, d$  lie in another. This can also be described as follows. Assume  $V$  is a finite subset of  $\mathbb{R}$ . Then the *support* of any part  $A \subseteq V$  is the closed interval  $supp A := [a, a'] \subset \mathbb{R}$  where  $a, a'$  are the minimum and maximum elements of  $A$ , respectively. A partition  $\mathcal{S}$  of  $V$  is noncrossing if, for any two parts  $A, B \in \mathcal{S}$ , one of the two sets is disjoint from the support of the other.

An important easy observation is the following.

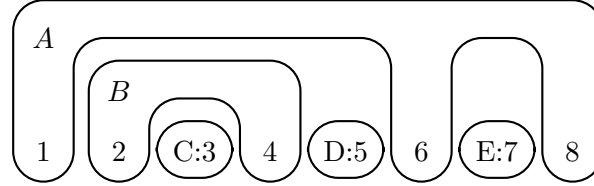
**Proposition 1.1.** *Given any noncrossing partition  $\mathcal{S}$  of  $V$ , another noncrossing partition of  $V$  can be given by taking the union of arbitrary noncrossing partitions of every part of  $\mathcal{S}$ .*

*Proof.* Given any two parts of  $\mathcal{S}$  the support of one, say  $A$ , is disjoint from the other, say  $B$ . Then the support of any part of  $A$  is disjoint from any part of  $B$ .  $\square$

This observation implies that it is very easy to determine how one part of a noncrossing partition can be split into two parts. However, the conditions for the converse operation are not immediate.

**Definition 1.2.** Given a noncrossing partition  $\mathcal{S}$  of a totally ordered set, which pairs of parts can be merged so that the resulting partition is still noncrossing? Two parts will be called *adjacent* if they have this property.

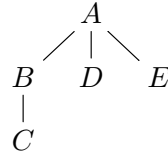
**Example 1.3.** Take the partition of  $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$  into five parts  $A = \{1, 6, 8\}$ ,  $B = \{2, 4\}$ ,  $C = \{3\}$ ,  $D = \{5\}$ ,  $E = \{7\}$ . This is a noncrossing partition.



Of the  $\binom{5}{2} = 10$  pairs of parts, five are adjacent and five are not:

- (1)  $E$  is not adjacent to  $B, C$  or  $D$ . The reason is that the union of  $E$  with  $B, C$  or  $D$  would contain 6 in its support and would thus cross  $A$ .
- (2)  $C, D$  are not adjacent since  $C \cup D$  crosses  $B$ .
- (3)  $A, C$  are not adjacent since  $A \cup C$  crosses  $B$
- (4)  $B, D$  are adjacent.
- (5) The other four pairs are adjacent and ordered:  $(B, A)$ ,  $(D, A)$ ,  $(C, B)$ ,  $(E, A)$ . In each of these pairs, the support of the first part is contained in the support of the second.

The parts of a noncrossing partition have two partial orderings. The first is by inclusion of support. We call this the *vertical ordering*. In the example we have  $C < B < A$ ,  $D < A$  and  $E < A$  in the vertical ordering. The Hasse diagram of the example is thus given by:



We will say that the part  $A$  of a noncrossing partition *covers* part  $B$  if  $A$  is directly above  $B$  in this Hasse diagram. In other words,  $\text{supp } B \subset \text{supp } A$  and there is no part  $C$  so that  $\text{supp } B \subset \text{supp } C \subset \text{supp } A$ .

**Lemma 1.4.** *If two parts  $A, B$  of a noncrossing partition are adjacent in the Hasse diagram of the partition then they are adjacent. Partially conversely, if there is another part  $C$  so that  $\text{supp } B \subset \text{supp } C \subset \text{supp } A$  then  $A, B$  are not adjacent.*

*Proof.* Suppose that  $A$  covers  $B$ . Then  $\text{supp}(A \cup B) = \text{supp } A$ . Given any other part  $C$ , there are three cases. (1) If  $A, C$  have disjoint support then  $A \cup B$  does not cross  $C$ . (2) If  $\text{supp } A \subset \text{supp } C$  then  $\text{supp } A$  does not meet  $C$ . Since  $\text{supp}(A \cup B) = \text{supp } A$ , the parts  $A \cup B$  and  $C$  do not cross. (3) If  $\text{supp } C \subset \text{supp } A$  then  $A$  is disjoint from  $\text{supp } C$  and either (3a)  $B, C$  have disjoint support or (3b)  $\text{supp } C \subset \text{supp } B$  since the remaining case (3c)  $\text{supp } B \subset \text{supp } C$  is excluded by assumption. In both subcases,  $\text{supp } C$  is disjoint from  $B$  as well as  $A$ . So,  $\text{supp } C$  is disjoint from  $A \cup B$  making them noncrossing. Therefore,  $A \cup B$  does not cross any other part of the partition.

For the partial converse, one sees immediately that  $A \cup B$  and  $C$  are crossing.  $\square$

When two parts  $A, B$  of a noncrossing partition have disjoint supports, they have a *lateral ordering* given by saying  $A$  is to the left of  $B$  if every element of  $A$  are less than every element of  $B$  in the total ordering of  $V$ . In the example,  $A$  covers three parts ordered laterally as  $B, D, E$ . However,  $B, D$  are adjacent but  $D, E$  are not. This is because there is an element of  $A$  between  $D$  and  $E$  in the lateral ordering.

We say that two parts  $A, B$  of a noncrossing partition  $\mathcal{S}$  are *parallel* if either

- (1)  $A, B$  are maximal or
- (2)  $A, B$  are covered by the same part  $C$  and no element of  $C$  lies between  $A$  and  $B$  in the lateral ordering. Equivalently,  $C$  is disjoint from the support of  $A \cup B$ .

Note that, in the second case, any other part  $D$  above  $A$  and  $B$  must also be above  $C$ . So,  $\text{supp}(A \cup B) \subseteq \text{supp} C$  is disjoint from  $D$ .

We say that a set of pairwise parallel parts of  $\mathcal{S}$  is *complete* if no parallel parts can be added to the set. A complete set of pairwise parallel parts will be called a *parallel set*.

**Proposition 1.5.** *Two parts of a noncrossing partition are adjacent if and only if the two parts are either parallel or one covers the other.*

*Proof.* The lemma takes case of the case when  $A, B$  are related in the vertical ordering. So, we may assume that  $A, B$  are two parts with disjoint support.

If  $A, B$  are parallel then, for any other part  $C$ , there are three cases: (1)  $C$  is above both  $A$  and  $B$ . (2)  $C$  is below one of them or (3)  $C$  is unrelated to both by the vertical ordering. In Case (1), the support of  $A \cup B$  does not contain any element of  $C$  by definition of parallel. In the other two cases, the support of  $C$  is disjoint from  $A \cup B$ . So,  $A \cup B$  and  $C$  do not cross. This shows that parallel parts are adjacent.

Suppose conversely that  $A, B$  are adjacent with disjoint supports. Then we claim that any other part  $C$  which lies above one must lie above the other. Otherwise,  $C$  and  $A \cup B$  would cross. So, either  $A, B$  are both maximal or they are covered by the same part  $C$ . In the second case we have that  $C$  is disjoint from the support of  $A \cup B$  since  $A, B$  are adjacent. In either case,  $A, B$  are parallel.  $\square$

We put together all adjacent pairs of parts into a single set  $E(\mathcal{S})$  which we call the *edge set* of the noncrossing partition  $\mathcal{S}$ . Formally, we define a (directed) *edge* to be a vector  $A - B \in \mathbb{Z}\mathcal{S}$  where  $A, B$  are distinct parts and  $\mathbb{Z}\mathcal{S}$  is the free abelian group generated by  $\mathcal{S}$ . We define  $E(\mathcal{S})$  to be the set of all edges  $A - B$  where either

- (1)  $A, B$  are parallel parts of  $\mathcal{S}$  or
- (2)  $A$  covers  $B$ .

**1.2. Binary forests.** Although graphs have vertices and edges, when we say that “ $G$  is a graph on a set  $S$ ” we mean that  $G$  is the set of edges of a graph and that  $S$  is its set of vertices. We define an *edge* (or *edge vector*) to be an element of the free abelian group  $\mathbb{Z}S$  of the form  $w - v$ . This is an edge from  $v$  to  $w$ .

By a (directed) *forest* on a finite set  $S$  we mean any linearly independent set  $F$  of edges  $E_i = w_i - v_i \in \mathbb{Z}S$ . A forest of maximal size is called a *tree*. These notions are easily seen to be equivalent to the standard notions of directed graphs which are forests and spanning trees.

Any forest  $F$  on  $S$  gives a partial ordering on  $S$  by the condition that  $v \leq w$  if there is a directed path in the forest from  $v$  to  $w$  or, equivalently,  $w - v$  is a sum of elements of  $F$ . A tree  $T$  on  $S$  is *rooted* if  $S$  has a unique maximal element  $r$ . This element is called the *root* of the tree. The *root vector* will be  $* - r \in \mathbb{Z}S_+$  where  $S_+ = S \cup \{*\}$ . The elements of  $S$  are the *vertices* of the tree but the basepoint  $*$  is not a “vertex”. We call  $T_+ = T \cup \{* - r\} \subset \mathbb{Z}S_+$  an *augmented tree*. The tree  $T \subset \mathbb{Z}S$  will be called a *rooted tree*. Sometimes it will be convenient to include the root vector  $* - r$  and sometimes not.

For an edge  $w - v$  in a rooted tree,  $w$  is called the *parent* of  $v$  and  $v$  is called a *child* of  $w$ . Note that each vertex of an augmented tree, including the root, will have exactly one parent.

We continue to assume that  $V$  is a finite totally ordered set. We will put a second “vertical” partial order on  $V$  which for notational convenience we write as follows. Let

$$V = \{v_1, \dots, v_n\}$$

The lateral ordering is given by the indices. For example, if we write  $v_i > v_j$ ,  $i < j$  we mean that  $v_i$  is above and to the left of  $v_j$ .

**Definition 1.6.** We define an *binary tree* on  $V$  to be a rooted tree  $T$  on  $V$  so that the induced partial ordering on  $V$  has the following additional properties.

- (1) If  $v_j - v_i \in T$  then  $v_k < v_i$  for any  $k$  between  $i$  and  $j$ .
- (2) Every  $v_i$  has at most two children.
- (3) If  $v_i$  has two children  $v_j, v_k$  then  $i$  lies between  $j$  and  $k$ .

Note that, by (1), all edges are either to the left of the root or to the right of the root. Therefore, if we remove the root  $r = v_k$  and edges connected to it, we will obtain two rooted binary trees, one on the set  $\{v_1, \dots, v_{k-1}\}$  and the other on the set  $\{v_{k+1}, \dots, v_n\}$  (unless  $k = 1$  or  $n$  in which case one of these two sets is empty).

Given a noncrossing partition  $\mathcal{S}$  on  $V$ , every parallel subset  $P$  of  $\mathcal{S}$  is totally ordered. So, it makes sense to talk about a binary tree on  $P$ .

**Definition 1.7.** A *binary forest* on a noncrossing partition  $\mathcal{S}$  on  $V$  is define to be the union of binary trees, one on each parallel subset of  $\mathcal{S}$ .

For example, if the parts of  $\mathcal{S}$  have disjoint support then a binary forest on  $\mathcal{S}$  is the same as a binary tree on  $\mathcal{S}$ . In other cases, we need to add edges to the forest to get a tree.

**Proposition 1.8.** *Let  $F$  be a binary forest on a noncrossing partition  $\mathcal{S}$ . Then  $F \subseteq E(\mathcal{S})$  and there exists a unique rooted tree  $T$  on  $\mathcal{S}$  so that  $F \subseteq T \subseteq E(\mathcal{S})$ .*

*Proof.* Since  $E(\mathcal{S})$  contains all edges whose endpoints are parallel,  $F$  is contained in  $E(\mathcal{S})$ .

The root of the parallel set of all maximal elements of  $\mathcal{S}$  must be the root of  $T$  since no edges in  $E(\mathcal{S})$  start at a maximal part by definition. For every other parallel set of  $\mathcal{S}$ , we need to add an edge starting at its root. This edge must point to the part which covers the parallel set. After adding these required edges, we have the unique rooted tree  $T \subseteq E(\mathcal{S})$  containing  $F$ .  $\square$

*Remark 1.9.* We call  $T$  the *rooted tree generated by  $F$* . Then  $T$  is a basis for the kernel of the augmentation map  $\varepsilon : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}$  given by  $\varepsilon(A) = 1$  for every  $A \in \mathcal{S}$ .

**1.3. Binary forests as morphisms.** Suppose that  $\mathcal{S}, \mathcal{R}$  are noncrossing partitions of  $V$  and  $\mathcal{S}$  is a refinement of  $\mathcal{R}$ . Then we have an epimorphism of sets  $\pi : \mathcal{S} \twoheadrightarrow \mathcal{R}$  sending each part of  $\mathcal{S}$  to the unique part of  $\mathcal{R}$  which contains it. Each  $W \in \mathcal{R}$  is totally ordered being a subset of  $V$ . And  $\pi^{-1}(W) \subseteq \mathcal{S}$  is a noncrossing partition of  $W$ . Therefore, it makes sense to talk about a binary forest  $F_W$  on  $\pi^{-1}(W)$  and the rooted tree  $T_W$  that it generates. We define a *cluster morphism*  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  of noncrossing partitions to be the union of a set of rooted trees  $T = \coprod T_W$  chosen in this way. The nomenclature will be explained later.

We need the following relative version of the set  $E(\mathcal{S})$ .

$$E(\mathcal{S}, \mathcal{R}) := \coprod_{W \in \mathcal{R}} E(\pi^{-1}(W))$$

Then  $E(\mathcal{S}, \mathcal{R})$  is a subset of the kernel of the linear epimorphism  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  induced by  $\pi$ . Applying the discussion of the previous subsection to each noncrossing partition  $\pi^{-1}(W)$  separately, we get the following.

**Proposition 1.10.** *The edges (elements) of any cluster morphism  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  lie in  $E(\mathcal{S}, \mathcal{R})$  and form a basis for the kernel of  $\pi_*$ . In particular,  $|T| = rk \mathcal{R} - rk \mathcal{S}$ .  $\square$*

**Lemma 1.11.** *Let  $\mathcal{R}, \mathcal{S}$  be as above. Then  $E(\mathcal{S}, \mathcal{R}) = E(\mathcal{S}) \cap \ker \pi_*$ . In particular,  $E(\mathcal{S}, \mathcal{R}) \subset E(\mathcal{S})$ .*

*Proof.* Suppose that  $A, B$  are parts of  $\mathcal{S}$  which lie in one part  $W \in \mathcal{R}$ . Then, it follows from Proposition 1.1 that  $A, B$  are adjacent as parts of the noncrossing partition  $\pi^{-1}(W)$  of  $W$  if and only if they are adjacent as parts of  $\mathcal{S}$ . Therefore,  $A - B$  lies in  $E(\mathcal{S}, \mathcal{R})$  if and only if it lies in  $E(\mathcal{S})$ . The lemma follows.  $\square$

**Proposition 1.12.** *Let  $\mathcal{R}, \mathcal{S}$  be as above and suppose  $\mathcal{R}$  is a refinement of  $\mathcal{Q}$ . Then  $E(\mathcal{S}, \mathcal{R}) = E(\mathcal{S}, \mathcal{Q}) \cap \ker \pi_*$ . In particular,  $E(\mathcal{S}, \mathcal{R})$  is a subset of  $E(\mathcal{S}, \mathcal{Q})$ .  $\square$*

A small category is defined to be a set of objects, a set of morphisms and a law of composition. We come to the last and most difficult part of the definition.

**1.4. Composition of cluster morphisms.** We define the *noncrossing partition category*  $\mathcal{P}(n)$  to be the category whose *objects* are the noncrossing partitions of  $\{1, \dots, n\}$ , whose *morphisms*  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  are the cluster morphisms defined above with composition defined below assuming two theorem which we will prove later.

**Definition 1.13.** Let  $\mathcal{R}, \mathcal{S}$  be objects of  $\mathcal{P}(n)$  so that  $\mathcal{S}$  is a refinement of  $\mathcal{R}$ . We define two edges  $E, E' \in E(\mathcal{S}, \mathcal{R})$  to be *compatible* if there is a cluster morphism from  $\mathcal{R}$  to  $\mathcal{S}$  which contains both of them.

**Theorem 1.14.** *A subset  $T$  of  $E(\mathcal{S}, \mathcal{R})$  gives a cluster morphism  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  if and only if the elements of  $T$  are pairwise compatible and  $T$  is maximal with this property.*

**Definition 1.15.** Let  $\mathcal{Q}, \mathcal{R}, \mathcal{S}$  be objects of  $\mathcal{P}(n)$  so that  $\mathcal{S}$  is a refinement of  $\mathcal{R}$  and  $\mathcal{R}$  is a refinement of  $\mathcal{Q}$ . Let  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  be a cluster morphism. Recall that  $T \subset E(\mathcal{S}, \mathcal{R}) \subseteq E(\mathcal{S}, \mathcal{Q})$ . We define  $E_T(\mathcal{S}, \mathcal{Q})$  to be the set of all element of  $E(\mathcal{S}, \mathcal{Q})$  which are compatible with the elements of  $T$  but not contained in  $T$ .

**Theorem 1.16.** *The linear map  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  induces a bijection  $E_T(\mathcal{S}, \mathcal{Q}) \cong E(\mathcal{R}, \mathcal{Q})$ . Furthermore,  $E, E'$  are compatible edges in  $E_T(\mathcal{S}, \mathcal{Q})$  if and only if the corresponding elements of  $E(\mathcal{R}, \mathcal{Q})$  are compatible.*

Let

$$\sigma_T : E(\mathcal{R}, \mathcal{Q}) \rightarrow E_T(\mathcal{S}, \mathcal{Q})$$

be the inverse of the bijection given by the theorem.

**Definition 1.17.** The composition of cluster morphisms  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  and  $[S] : \mathcal{Q} \rightarrow \mathcal{R}$  is given by:

$$[S] \circ [T] = [T \cup \sigma_T S] : \mathcal{Q} \rightarrow \mathcal{S}$$

We verify that this is a cluster morphism. By Theorem 1.14,  $S$  is a maximal compatible subset of  $E(\mathcal{R}, \mathcal{Q})$ . By Theorem 1.16,  $\sigma_T(S)$  is a maximal compatible subset of  $E_T(\mathcal{S}, \mathcal{Q})$ . This is equivalent to the statement that  $T \cup \sigma_T(S)$  is a maximal compatible subset of  $E(\mathcal{S}, \mathcal{Q})$  which implies that  $[T \cup \sigma_T(S)]$  is a cluster morphism  $\mathcal{Q} \rightarrow \mathcal{S}$ .

Next, we will verify that composition of morphisms is associative. Let

$$\mathcal{P} \xrightarrow{[R]} \mathcal{Q} \xrightarrow{[S]} \mathcal{R} \xrightarrow{[T]} \mathcal{S}$$

be three composable cluster morphisms. Then

$$\begin{aligned} [T] \circ ([S] \circ [R]) &= [T] \circ [R \cup \sigma_R S] = [R \cup \sigma_R S \cup \sigma_{R \cup \sigma_R S} T] \\ ([T] \circ [S]) \circ [R] &= [S \cup \sigma_S T] \circ [R] = [R \cup \sigma_R S \cup \sigma_R \sigma_S T] \end{aligned}$$

So, it suffices to show that  $\sigma_R \circ \sigma_S = \sigma_{R \cup \sigma_R S}$ . But, the first map is the composition of two bijections

$$E(\mathcal{Q}, \mathcal{P}) \xrightarrow{\sigma_S} E_S(\mathcal{R}, \mathcal{P}) \xrightarrow{\sigma_R} E_{R \cup \sigma_R S}(\mathcal{S}, \mathcal{P})$$

whose inverses are given by the projection maps  $\mathbb{Z}\mathcal{S} \twoheadrightarrow \mathbb{Z}\mathcal{R} \twoheadrightarrow \mathbb{Z}\mathcal{Q}$ . And, the second map is the bijection  $E(\mathcal{Q}, \mathcal{P}) \cong E_{R \cup \sigma_R S}(\mathcal{S}, \mathcal{P})$  whose inverse is given by the projection map  $\mathbb{Z}\mathcal{S} \twoheadrightarrow \mathbb{Z}\mathcal{Q}$ . So, these two maps agree.

Therefore, the definition of the category of noncrossing partitions will be complete when we prove Theorems 1.14 and 1.16.

## 2. COMPATIBILITY OF EDGE SETS

The purpose of this section is to prove Theorem 1.14 and Theorem 1.16, the two theorems needed to define composition of cluster morphisms. The same idea is also needed in the proof that the classifying space of the category of noncrossing partitions is  $CAT(0)$ . We will first do the basic case of Theorem 1.14 to show that binary trees are maximal compatible sets of edges.

**2.1. Binary trees as maximal compatible sets.** By definition a binary tree  $T$  has a unique root, say  $v_k$ . We will augment this tree with the root vector  $* - v_k$ . If  $V = \{v_1, \dots, v_n\}$  is a finite totally set (the set of vertices), let  $V_+ = V \cup \{*\}$ . Define  $E(V)$  to be the subset of the free abelian group  $\mathbb{Z}V \cong \mathbb{Z}^n$  consisting of the  $n(n-1)$  vectors  $v_j - v_i$  for  $i \neq j$  which we call *edge vectors*. Let  $G(V) \subset \mathbb{Z}V_+ \cong \mathbb{Z}^{n+1}$  be the union of  $E(V)$  and the  $n$  vectors  $* - v_k$  which we call *root vectors*. So,  $|G(V)| = n^2$ . We define an *augmented binary tree*  $T_+$  on  $V$  to be a subset of  $G(V)$  with  $n$  elements:

- (1) the edge vectors  $v_j - v_i$  of the  $n-1$  directed edges  $v_i \rightarrow v_j$  of the tree and
- (2) the (unique) root vector  $* - v_k$  of the root  $v_k$  of the tree.

As before the *support* of  $E = v_i - v_j$  is the closed interval in  $\mathbb{R}$  with endpoints  $i, j$ . The support of a root vector  $* - v_k$  is  $[k, \infty)$  and the length of the root vector is infinite.

The definition we want is: “Two elements of  $G(V)$  are compatible if there exists a binary tree which contains both of them.” For example,  $v_1 - v_2$  is compatible with  $* - v_1$  but not with  $* - v_2$  since the inclusion of the edge  $v_1 - v_2$  implies  $v_1 > v_2$  in the partial ordering given by the tree. So,  $v_1$  can be a root, but not  $v_2$ . The theorem we want is: “Binary trees are the same as maximal compatible subsets of  $G(V)$ .” To prove this we need a more precise statement.

**Definition 2.1.** *Compatibility* of pairs of elements of  $G(V)$  is defined by the following conditions.

- (1) A root vector  $* - v_k$  is not compatible with any other root vector.
- (2) A root vector  $* - v_k$  and an edge  $E = v_j - v_i$  are compatible if and only if either  $k = j$  or  $k$  lies outside the support of  $E$ .
- (3) Given two edges  $E_1$  and  $E_2$  there are several possibilities:



- (a) (noncrossing condition) If  $E_1, E_2$  have distinct endpoints, they are compatible if and only if either their supports are disjoint or the support of one is contained in the interior of the support of the other.
- (b) If the intersection of the supports of  $E_1, E_2$  is one point  $v_j$  then they are compatible if and only if they do not both point away from  $v_j$ .
- (c) If two edges share one endpoint  $v_k$  and the support of one is contained in the support of the other then they are compatible if and only if they have different lengths and the longer edge points away from  $v_k$  and the shorter edge points towards  $v_j$ .

Conditions (3b), (3c) can be combined into one condition as follows.

- (3bc) Every vertex  $v_j$  has at most one left child, at most one right child and at most one parent. Furthermore a child on the same side as a parent must be strictly closer to  $v_j$  than the parent.

It follows from the definition of a binary tree that the edges and root vector of a binary tree on  $V$  form a pairwise compatible subset of  $G(V)$ . We will prove the converse:

**Theorem 2.2.** *Any maximal pairwise compatible subset of  $G(V)$  has  $n$  elements and consists of the edges and root vector of a binary tree on  $V$ .*

Some immediate consequences of this theorem are the following.

- Corollary 2.3.**
- (1) *Every compatible pair of elements of  $G(V)$  lie in an augmented binary tree on  $V$ .*
  - (2) *The root vector and edges of any binary tree on  $V$  form a maximal compatible subset of  $G(V)$ .*
  - (3) *Any maximal compatible subset of  $E(V)$  has  $n-1$  elements and consists of the edges of a binary tree on  $V$ .*

*Proof.* (1) Any compatible set is contained in a maximal compatible set which forms an augmented binary tree by the theorem.

(2) The parts of an augmented binary tree are pairwise compatible. So, they form a subset of a maximal pairwise compatible set which, by the theorem, must be the set we started with.

(3) Any maximal compatible subset  $T$  of  $E(V)$  is contained in a maximal compatible subset of  $G(V)$  which contains one root vector. Removing the root vector must give  $T$ .  $\square$

We prove Theorem 2.2 by induction on the size of  $V$  using the root.

**Lemma 2.4.** *Any maximal pairwise compatible subset of  $G(V)$  contains a root vector. Equivalently, for any pairwise compatible subset  $S$  of  $E(V)$  there is at least one root vector compatible with all elements of  $S$ .*

*Proof.* Let  $S \subset E(V)$  be a compatible set and let  $E = v_{j_1} - v_{j_0} \in S$  be of greatest length. By symmetry we may assume  $j_1 > j_0$ . Thus  $v_{j_1}$  is a right parent of  $v_{j_0}$ . For each  $i$  let  $v_{j_{i+1}}$  be the right parent of  $v_{j_i}$  if it exists. Then  $j_{i+1} > j_i$ . So, this sequence eventually stops at some  $v_{j_m}$ .

Claim 1:  $v_{j_m}$  is maximal in the partial ordering induced by  $S$ .

Claim 2:  $S$  has no edge  $E'$  so that  $v_{j_m}$  lies in the interior of the support of  $E'$ .

*Proof:* We prove these at the same time. By construction,  $v_{j_m}$  has no right parent. So, if Claim 1 fails then  $v_{j_m}$  has a left parent  $v_k$ . So,  $E' = v_k - v_{j_m} \in S$ . Thus, if either claim

fails,  $S$  will have an edge  $E'$  with left endpoint  $v_k$  with  $k < j_m$  and right endpoint  $v_\ell$  with  $\ell \geq j_m$  so that  $E' \neq v_{j_m} - v_{j_{m-1}}$ .

Since  $E$  has maximal length and  $j_1 \leq j_m$ , we must have  $k > j_0$ . By the noncrossing condition,  $v_k$  cannot fall between two vertices in the sequence  $v_{j_i}$ . Therefore,  $v_k = v_{j_i}$  for some  $j_i$  where  $1 \leq i \leq m-1$ . Since  $v_k$  cannot have two parents,  $v_\ell$  must be a right child of  $v_k$ . But then,  $v_k = v_{j_i}$  has a right parent  $v_{j_{i+1}}$  at least as close as its right child  $v_\ell$  which is a contradiction. So,  $E'$  does not exist and both claims hold.

Claims 1 and 2 imply that  $* - v_{j_m}$  is compatible with every element of  $S$ . □

*Proof of Theorem 2.2.* (This argument is also the beginning of the proof of Theorem 1.14.)

The theorem clearly holds for  $n = 1$ . So, suppose that  $n \geq 2$  and the theorem holds for  $n-1$ . Let  $S$  be a maximal compatible subset of  $G(V)$ . By the lemma,  $S$  contains a unique root vector  $* - v_k$ .

Case 1:  $k = 1$  or  $n$ . Then  $v_k$  has at most one child. Let  $S'$  be obtained from  $S$  by removing  $* - v_k$  and all edges with endpoint  $v_k$ . Since  $S'$  is a compatible subset of  $E(V')$ ,  $V' = V \setminus v_k$ , there is a root vector  $* - v_j \in G(V')$  compatible with  $S'$ . We can take  $v_j$  to be the child of  $v_k$  if one exists. Then  $S' \cup \{ * - v_j \}$  is contained in a maximal compatible subset  $T'_+$  of  $G(V')$  which, by induction, must be an augmented binary tree. Then the union of  $T'_+ \setminus \{ * - v_j \}$  with  $* - v_k$  and  $E = v_k - v_j$  will be a rooted binary tree which contains  $S$  and thus must be equal to  $S$  by maximality of  $S$ . So, the theorem holds in this case.

Case 2:  $1 < k < n$ . Then all edges in  $S$  are contained either in  $E(V')$  or  $E(V'')$  where  $V' = \{v_1, \dots, v_{k-1}\}$  and  $V'' = \{v_{k+1}, \dots, v_n\}$ . Let  $S'$  be the union of  $S \cap E(V')$  with a compatible root vector  $* - v_i$  where we take  $v_i$  to be the left child of  $v_k$  if one exists. Let  $S''$  be defined similarly with root vector  $* - v_j$ . By induction on  $n$ ,  $S', S''$  are contained in augmented binary trees  $T'_+ \subseteq G(V')$  and  $T''_+ \subseteq G(V'')$ . Then the union of  $T'_+ \setminus \{ * - v_i \}$  with  $T''_+ \setminus \{ * - v_j \}$  and  $* - v_k, v_k - v_i, v_k - v_j$  is an augmented binary tree on  $V$  which contains  $S$  and therefore must be equal to  $S$ .

So, the theorem holds in all cases. □

Since we now know that maximal compatible subsets of  $G(V)$  are augmented binary tree, the argument in this proof can be rephrased as follows. For any  $X \in G(V)$  we use the notation  $G_X(V)$  for the set of all elements of  $G(V)$  which are compatible with  $X$ .

**Corollary 2.5.** *Let  $R = * - v_k$  be a root vector in  $G(V)$  where  $V = \{v_1, \dots, v_n\}$ .*

- (1) *If  $k = 1$  or  $n$  then there is a bijection*

$$\sigma_R : G(V \setminus \{v_k\}) \rightarrow G_R(V)$$

*given by  $\sigma_R(E) = E$  for all  $E \in E(V \setminus R)$  and  $\sigma_R(* - v_j) = v_k - v_j$ . Furthermore,  $X, Y \in G(V \setminus \{v_k\})$  are compatible if and only if  $\sigma_R(X), \sigma_R(Y)$  are compatible in  $G_R(V)$ . When  $k = n = 1$ ,  $\sigma_R$  is a bijection between two empty sets.*

- (2) *If  $1 < k < n$  then there is a bijection*

$$\sigma_R : G(\{v_1, \dots, v_{k-1}\}) \amalg G(\{v_{k+1}, \dots, v_n\}) \rightarrow G_R(V)$$

*given by  $\sigma_R(E) = E$  for all edges  $E$  and  $\sigma_R(* - v_j) = v_k - v_j$  for all root vectors  $* - v_j$ . Furthermore,  $\sigma_T(X), \sigma_T(Y) \in G(V)$  are compatible if and only if  $X, Y$  lie in different blocks*

This can be rephrased as follows.

$\sigma_R(X)$  is the unique element of  $G_R(V)$  congruent to  $X$  modulo  $R$ .

**2.2. Proof of Theorem 1.14.** Let  $\mathcal{R}, \mathcal{S}$  be noncrossing partitions of  $V$  and suppose that  $\mathcal{S}$  is a refinement of  $\mathcal{R}$ . Then  $E(\mathcal{S}, \mathcal{R})$  is a disjoint union of *blocks*  $\mathcal{B}_{\alpha\beta}$  defined as follows.

Let  $\iota : \mathcal{S} \rightarrow \mathcal{R}$  be the mapping which sends each part of  $\mathcal{S}$  to the unique part of  $\mathcal{R}$  which contains it. For each  $W_\alpha \in \mathcal{R}$  let  $\mathcal{U}_\alpha = \iota^{-1}(W_\alpha) \subseteq \mathcal{S}$ . Then  $\mathcal{U}_\alpha$  is a noncrossing partition of  $W_\alpha \subseteq V$ . So,  $\mathcal{U}_\alpha$  is a disjoint union of parallel sets  $S_{\alpha\beta}$  of which one, say  $S_{\alpha 0}$ , is maximal. Every other  $S_{\alpha\beta}$  is covered by some part  $C_{\alpha\beta} \in \mathcal{U}_\alpha$ .

**Definition 2.6.** We define the *maximal block* associated to the maximal parallel set  $S_{\alpha 0}$  to be the set  $\mathcal{B}_{\alpha 0} = E(S_{\alpha 0}) \subseteq E(\mathcal{S}, \mathcal{R})$ . For  $\beta \neq 0$ , define the *block*  $\mathcal{B}_{\alpha\beta}$  associated to the parallel set  $S_{\alpha\beta}$  covered by  $C_{\alpha\beta}$  to be the image of the embedding

$$\psi : G(S_{\alpha\beta}) \hookrightarrow E(\mathcal{S}, \mathcal{R})$$

which sends each root vector  $* - X$  to the edge  $C_{\alpha\beta} - X$  and which is the inclusion map on  $E(S_{\alpha\beta})$ .

**Lemma 2.7.**  *$E(\mathcal{S}, \mathcal{R})$  is a disjoint union of blocks. Elements of different blocks are always compatible. Two elements of the same block are compatible if and only if the corresponding elements of  $G(S_{\alpha\beta})$  (resp.  $E(S_{\alpha 0})$ ) are compatible as pieces of augmented (resp. unaugmented) binary trees on the parallel set.*

*Proof.* This follows from the definitions of the words. □

*Proof of Theorem 1.14.* This follows immediately from Theorem 2.2 and the lemma above. Indeed a cluster morphism is defined to be a binary tree structure on every parallel set together with the edge from the root of that binary tree to the part which covers the parallel set in the case the parallel set is not maximal. By Theorem 2.2, such structures are maximal compatible subsets of the associated block in  $E(\mathcal{S}, \mathcal{R})$ . By the lemma, the maximal compatible subsets of  $E(\mathcal{S}, \mathcal{R})$  are disjoint unions of maximal compatible subsets of the blocks. Therefore, these maximal compatible sets are the cluster morphisms  $\mathcal{R} \rightarrow \mathcal{S}$ . □

**2.3. First case of Theorem 1.16.** Recall that, for  $E \in G(V)$ ,  $G_E(V)$  is the set of all  $X \in G(V)$  which are compatible with  $E$ . We analyzed the case when  $E$  is a root vector in Corollary 2.5. This translates into one case of Theorem 1.16:

Suppose that  $\mathcal{U} = \{X_1, \dots, X_n\}$  is a nonmaximal parallel set in a noncrossing partition  $\mathcal{S}$  and let  $Y$  be the part of  $\mathcal{S}$  which covers  $\mathcal{U}$ . Let  $\mathcal{R}$  be the noncrossing partition obtained from  $\mathcal{S}$  by merging the parts  $X_k, Y$  to get a new part  $Z = X_k \cup Y$ . Recall that  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  is the linear surjection sending  $X_k$  and  $Y$  to  $Z$  and all other parts of  $\mathcal{S}$  to the same part in  $\mathcal{R}$ . The kernel of  $\pi_*$  is the set of all integer multiples of the vector  $R = Y - X_k$ .

**Lemma 2.8.** *(root vector case) The linear map  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  induces a bijection  $E_{\mathcal{R}}(\mathcal{S}) \cong E(\mathcal{R})$  and this bijection preserves the relation of compatibility and non-compatibility.*

*Proof.* When  $X_k$  is merged with  $Y$ , the parallel set  $\mathcal{U}$  in  $\mathcal{S}$  is, in general, divided into two parallel sets in  $\mathcal{R}$ . There are exceptional cases when  $k = 1$  or  $n$  or  $n = 1$  analogous to trees listed in Corollary 2.5.

The set  $E(\mathcal{S})$  is a disjoint union of blocks and the block isomorphic to  $G(\mathcal{U})$  becomes  $G_{\mathcal{R}}(\mathcal{U})$  which is isomorphic to  $G(\mathcal{U}') \amalg G(\mathcal{U}'')$  as in the corollary and these are isomorphic to the corresponding blocks of  $\mathcal{R}$ . Since the basepoint  $* \in G(\mathcal{U}'')$  and  $* \in G(\mathcal{U}')$  both correspond to  $Z \in \mathcal{R}$ , the equation for the bijection  $\sigma_{\mathcal{R}}$  becomes

$$\sigma_{\mathcal{R}}(* - X_j) = \sigma_{\mathcal{R}}(Y - X_j) = X_k - X_j$$

In other words,  $\sigma_R(X)$  is congruent to  $X$  module  $R = Y - X_k$ . This is equivalent to the statement that  $\sigma_R^{-1} = \pi_*$ .

Since  $\sigma_R$  takes compatible pairs of elements to compatible pairs of elements in the case of trees,  $\pi_* = \sigma_R^{-1}$  preserves compatibility relations for noncrossing partitions since they are defined in terms of compatibility in the blocks that they lie in.  $\square$

Now we examine the case when  $E = X_b - X_a$  is an edge between two parallel parts of a noncrossing partition. As in the root vector case, the statement follows from the corresponding statement about compatible sets of edges for augmented binary trees.

The basic idea is that, when we build up an augmented binary tree starting with a fixed edge  $E = v_k - v_{k-1}$  of length one, this edge behaves like a single vertex which we will label  $v_k$ . An augmented binary tree  $T_+$  on  $V$  which contains  $E$  is equivalent to an augmented binary tree  $T'_+$  on  $V' = \{v_1, \dots, \widehat{v_{k-1}}, \dots, v_n\}$ , We take  $T_+$  to be the union of all elements of  $T'_+$  except for those of the form  $v_k - v_j$  for  $j < k$  which we replace with  $v_{k-1} - v_j \in T_+$ . (So that  $v_k$  does not have two left children in  $T_+$ .)

If  $T_+$  contains a fixed edge  $E = v_b - v_a$  or length  $b - a \geq 2$  then it will also contain a binary tree  $T''$  on the set  $V'' = \{v_{a+1}, \dots, v_{b-1}\}$  plus an edge from the root of  $T''$  to  $v_a$  since, by the noncrossing condition, no edge of  $T_+$  can go from inside the interval  $(a, b)$  to outside  $[a, b]$ . We treat this edge as corresponding to the root vector for  $T''$ . The rest of  $T_+$  will be equivalent to an augmented binary tree  $T'_+$  on  $V' = \{v_1, \dots, v_{a-1}, v_b, \dots, v_n\}$  with any edge  $v_b - v_j \in T'_+$  with  $j < a$  replaced with  $v_a - v_j \in T_+$ :

$$v_b - v_j \in T'_+ \mapsto v_a - v_j \in T_+$$

This analysis for one edge  $E$  in one tree  $T_+$  translates into the following correspondence for all possible components of the trees  $T_+, T'_+, T''$ . We note that  $T_+, T'_+, T''$  are arbitrary augmented binary trees on their vertex sets. (I.e., starting with any  $T_+$  containing  $E$ , we get  $T'_+, T''$  and conversely, starting with any  $T'_+, T''$  we get a unique  $T_+$  containing  $E$ .)

To do these cases simultaneously it is helpful to remember that  $G(V)$  has  $n^2$  elements. In particular,  $G(\emptyset) = \emptyset$ .

**Lemma 2.9.** *Suppose that  $E = v_b - v_a$  where  $1 \leq a < b \leq n$ . Then there is a bijection*

$$\sigma_E : G(V') \amalg G(V'') \rightarrow G_E(V)$$

where  $V' = \{v_1, \dots, v_{a-1}, v_b, v_{b+1}, \dots, v_n\}$  and  $V'' = \{v_{a+1}, \dots, v_{b-1}\}$  given by sending each  $E' \in G(V')$  to  $\sigma_E(E') = E' \in G_E(V)$  except for edges of the form  $v_b - v_j \in G(V')$  for  $j < a$  where we take

$$\sigma_E(v_b - v_j) = v_a - v_j.$$

For  $G(V'')$ , each edge  $E'' \in E(V'')$  is sent to itself and the root vector  $* - v_k$  is sent to

$$\sigma_E(* - v_k) = v_a - v_k.$$

Furthermore, the bijection  $\sigma_E$  has the property that  $X, Y$  in the domain of  $\sigma_E$  are compatible if and only if  $\sigma_E(X), \sigma_E(Y)$  are compatible in  $G_E(V)$ .

The analogous statement for  $E = v_a - v_b$  is the following.

**Lemma 2.10.** *Suppose that  $E = v_a - v_b$  where  $1 \leq a < b \leq n$ . Then there is a bijection*

$$\sigma_E : G(V') \amalg G(V'') \rightarrow G_E(V)$$

where  $V' = \{v_1, \dots, v_a, v_{b+1}, \dots, v_n\}$  and  $V'' = \{v_{a+1}, \dots, v_{b-1}\}$  given by sending each  $E' \in G(V')$  to  $\sigma_E(E') = E' \in G_E(V)$  except for edges of the form  $v_a - v_j \in G(V')$  for  $j > b$  where we take

$$\sigma_E(v_a - v_j) = v_b - v_j.$$

For  $G(V'')$ , each edge  $E'' \in E(V'')$  is sent to itself and the root vector  $* - v_k$  is sent to

$$\sigma_E(* - v_k) = v_b - v_k.$$

Furthermore, the bijection  $\sigma_E$  has the property that  $X, Y$  in the domain of  $\sigma_E$  are compatible if and only if  $\sigma_E(X), \sigma_E(Y)$  are compatible in  $G_E(V)$ .

*Proof.* In the discussion before the statements we started with an arbitrary augmented binary tree on  $V$  with a fixed edge  $E$  and constructed augmented binary trees on  $V', V''$  which were also arbitrary. Two compatible elements of  $G_E(V)$  can be extended to a tree. Passing to the corresponding trees on  $V', V''$ , we see that the corresponding edges are compatible.  $\square$

These statements are equivalent to the following special case of Theorem 1.16. Suppose that  $\mathcal{S}$  is a refinement of  $\mathcal{Q}$ . We recall that, any edge  $E \in E(\mathcal{S}, \mathcal{Q})$  has the form  $E = X_b - X_a$  where either  $X_b$  covers  $X_a$  or  $X_a, X_b$  form part of a parallel set  $\mathcal{U} = \{X_1, \dots, X_m\} \subseteq \iota^{-1}(W)$  in the inverse image  $\iota^{-1}(W) \subseteq \mathcal{S}$  of one part  $W \in \mathcal{Q}$ . Recall that, in the second case,  $E(\mathcal{S}, \mathcal{Q})$  contains a block  $\mathcal{B}$  isomorphic to  $G(\mathcal{U})$  (or to  $E(\mathcal{U})$  when  $\mathcal{U}$  is maximal).

The partition  $\mathcal{R}$  obtained from  $\mathcal{S}$  by merging the parts  $X_a, X_b$  together is noncrossing and  $\mathcal{R}$  is a refinement of  $\mathcal{Q}$ . Recall that  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  is the linear surjection sending  $X_a$  and  $X_b$  to  $Z = X_a \cup X_b$  and all other parts  $Y \in \mathcal{S}$  to the same part  $Y \in \mathcal{R}$ . The kernel of  $\pi_*$  is the set of all integer multiples of the vector  $E = X_a - X_b$ .

**Lemma 2.11.** *When  $rk \mathcal{R} = 1 + rk \mathcal{S}$ , the linear map  $\pi_* : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{R}$  induces a bijection  $E_E(\mathcal{S}, \mathcal{Q}) \cong E(\mathcal{R}, \mathcal{Q})$  and this bijection preserves the relation of compatibility and non-compatibility.*

*Proof.* Without loss of generality we may assume that  $\mathcal{Q}$  has only one part since parts of  $\mathcal{S}$  and  $\mathcal{R}$  in different parts of  $\mathcal{Q}$  are unrelated. Then  $E(\mathcal{R}, \mathcal{Q}) = E(\mathcal{R})$  and  $E_E(\mathcal{S}, \mathcal{Q}) = E_E(\mathcal{S})$ . The case when  $X_b$  covers  $X_a$  was settled in Lemma 2.8. So, suppose  $X_a, X_b$  are parallel.

Any block  $\mathcal{B}' \subseteq E(\mathcal{S})$  other than  $\mathcal{B}$  is a block of both  $E_E(\mathcal{S})$  and  $E(\mathcal{R})$  and  $\pi_*(\mathcal{B}') = \mathcal{B}'$ . Let  $\coprod \mathcal{B}'$  be the union of these other blocks in both sets. Then, it suffices to consider the complement of  $\coprod \mathcal{B}'$  in  $E_E(\mathcal{S})$  and  $E(\mathcal{R})$ . These are the subsets corresponding to the block  $\mathcal{B}$  in  $E(\mathcal{S})$  which is isomorphic to  $G(\mathcal{U})$ ,  $\mathcal{U} = \{X_1, \dots, X_m\}$ . The corresponding subset of  $E_E(\mathcal{S})$  is

$$G_E(\mathcal{U}) = G(\mathcal{U}) \cap E_E(\mathcal{S}) = E_E(\mathcal{S}) \setminus \coprod \mathcal{B}'$$

By the above two lemmas, this is in bijection with the union of the two blocks in  $E(\mathcal{R})$  given by the parallel sets

$$\mathcal{U}' = \{X_1, \dots, X_{a-1}, Z, X_{b+1}, \dots, X_m\} \text{ and } \mathcal{U}'' = \{X_{a+1}, \dots, X_{b-1}\}$$

in  $\mathcal{R}$  to  $G_E(\mathcal{U})$ . Furthermore, this bijection sends each edge  $X - Y$  to the same edge  $X - Y$  except in the case when one of the parts  $X, Y$  is equal to  $Z = X_a \cup X_b$  in which case it is replaced by  $X_a$  or  $X_b$  whichever produces an edge compatible with  $E$ .

In all cases, the bijection  $\sigma_E$  of Lemmas 2.9 and 2.10 takes elements of  $E(\mathcal{R})$  to elements in their inverse image in  $E_E(\mathcal{S})$ . So,  $\sigma_E = \pi_*^{-1}$ . Since  $\sigma_E$  preserves compatibility, so does  $\pi_*$ .  $\square$

**2.4. Proof of Theorem 1.16.** The proof of Theorem 1.16 is now an easy induction on the difference in ranks  $rk \mathcal{R} - rk \mathcal{S}$ . By Lemma 2.11, the theorem holds when this difference is 1. So, suppose that  $\mathcal{S}$  is a refinement of  $\mathcal{R}$  and  $\mathcal{R}$  is a refinement of  $\mathcal{Q}$  and  $rk \mathcal{R} - rk \mathcal{S} = k \geq 2$ .

Let  $T = \{E_1, \dots, E_k\}$  be a cluster morphism  $\mathcal{R} \rightarrow \mathcal{S}$ . Then  $T \subset E(\mathcal{S}, \mathcal{Q})$ . Suppose that  $E_1 = Y - X$  and let  $\mathcal{P}$  be obtained from  $\mathcal{S}$  by fusing  $X, Y$  together to a single part  $Z = XUY$ . Then  $\mathcal{P}$  is a refinement of  $\mathcal{R}$  and, by Lemma 2.11, the linear mapping  $\pi_*^{S\mathcal{P}} : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{P}$  induces bijections:

$$\pi_*^{S\mathcal{P}} : E_{E_1}(\mathcal{S}, \mathcal{Q}) \cong E(\mathcal{P}, \mathcal{Q}), \quad \pi_*^{S\mathcal{P}'} : E_{E_1}(\mathcal{S}, \mathcal{R}) \cong E(\mathcal{P}, \mathcal{R})$$

where the second bijection is the restriction of the first to  $E_{E_1}(\mathcal{S}, \mathcal{R}) \subseteq E_{E_1}(\mathcal{S}, \mathcal{Q})$ .

Since  $[T] : \mathcal{R} \rightarrow \mathcal{S}$  is a cluster morphism,  $E_2, \dots, E_k$  are compatible with  $E_1$  and with each other. So,  $E_2, \dots, E_k \in E_{E_1}(\mathcal{S}, \mathcal{R}) \subseteq E_{E_1}(\mathcal{S}, \mathcal{Q})$  which map to compatible elements  $\pi_*^{S\mathcal{P}}(E_i) \in E(\mathcal{P}, \mathcal{R}) \subseteq E(\mathcal{P}, \mathcal{Q})$ . Since  $rk \mathcal{R} - rk \mathcal{P} = k - 1$ , this means that  $S = \{\pi_*^{S\mathcal{P}}(E_2), \dots, \pi_*^{S\mathcal{P}}(E_k)\}$  gives a cluster morphism  $[S] : \mathcal{P} \rightarrow \mathcal{R}$ .

Since  $\pi_*^{S\mathcal{P}}$  sends compatible elements of  $E_{E_1}(\mathcal{S}, \mathcal{Q})$  to compatible elements of  $E(\mathcal{P}, \mathcal{Q})$ , it induces a bijection

$$E_T(\mathcal{S}, \mathcal{Q}) \cong E_S(\mathcal{P}, \mathcal{Q})$$

By induction on  $k$ , the linear map  $\pi_*^{\mathcal{P}\mathcal{R}} : \mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{R}$  induces a bijection

$$\pi_*^{\mathcal{P}\mathcal{R}} : E_S(\mathcal{P}, \mathcal{Q}) \cong E(\mathcal{R}, \mathcal{Q})$$

Composing these we get a bijection

$$E_T(\mathcal{S}, \mathcal{Q}) \cong E(\mathcal{R}, \mathcal{Q})$$

induced by the composite linear mapping  $\pi^{S\mathcal{R}} = \pi^{\mathcal{P}\mathcal{R}} \circ \pi^{\mathcal{R}\mathcal{S}} : \mathbb{Z}\mathcal{S} \rightarrow \mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{R}$ . Since  $\pi^{\mathcal{P}\mathcal{R}}$  and  $\pi^{\mathcal{R}\mathcal{S}}$  both preserve compatibility, so does their composite  $\pi^{S\mathcal{R}}$ .

This proves Theorem 1.16 for all  $k$  and complete the proof that composition of cluster morphisms is well-defined and associative.

### 3. CUBE COMPLEXES

We will show that the classifying space of the category of noncrossing partitions is a locally  $CAT(0)$  cube complex and therefore a  $K(\pi, 1)$ . This follows from general sufficient conditions on a cubical category which we will describe.

**3.1. Cubical categories.** The basic model for a category whose geometric realization is an  $n$  cube is the category  $\mathcal{I}^n$  where  $\mathcal{I}$  is the poset category with two objects  $0, 1$  and one nonidentity morphism  $0 \rightarrow 1$  corresponding to the relation  $0 < 1$ . Recall that a poset  $P$  can be taken to be the set of objects of a category  $\mathcal{C}(P)$  with a single morphism  $A \rightarrow B$  whenever  $A \leq B$  and no morphisms  $A \rightarrow B$  otherwise.

The product category  $\mathcal{I}^n$  is also a poset category. The set of objects is the set of all vectors  $x \in \mathbb{R}^n$  so that every coordinate is either 0 or 1. We have  $x \leq y$  and thus a unique morphism  $x \rightarrow y$  iff  $x_i \leq y_i$  for  $i = 1, \dots, n$ . It is very easy to see that the geometric realization of the category  $\mathcal{I}^n$  is canonically homeomorphic to the  $n$ -cube  $[0, 1]^n$ . We will define a ‘‘cubical category’’ to be a graded category in which the factorization category of any morphism of degree  $n$  is isomorphic to  $\mathcal{I}^n$  for some  $n$  plus other conditions. To be consistent with the rest of the paper we will use the word ‘‘rank’’ instead of ‘‘degree.’’

**Definition 3.1.** Let  $f : A \rightarrow B$  be a morphism in a category  $\mathcal{C}$ . Then the *factorization category*  $Fac(f)$  is the category whose objects are *factorizations* of  $f$  given by triples  $(C, g, h)$  where  $C$  is an object of  $\mathcal{C}$  and  $g, h$  are morphisms  $g : A \rightarrow C, h : C \rightarrow B$  so that  $f = h \circ g$ .

$$A \xrightarrow{g} C \xrightarrow{h} B$$

A morphism  $(C, g, h) \rightarrow (C', g', h')$  is defined to be a morphism  $\phi : C \rightarrow C'$  so that  $g' = \phi \circ g$  and  $h = h' \circ \phi$ . There is a forgetful functor  $Fac(f) \rightarrow \mathcal{C}$  taking  $(C, g, h)$  to  $C$ . The morphism  $f$  is *irreducible* if, for any factorization  $f = h \circ g$ , either  $g$  or  $h$  is an isomorphism.

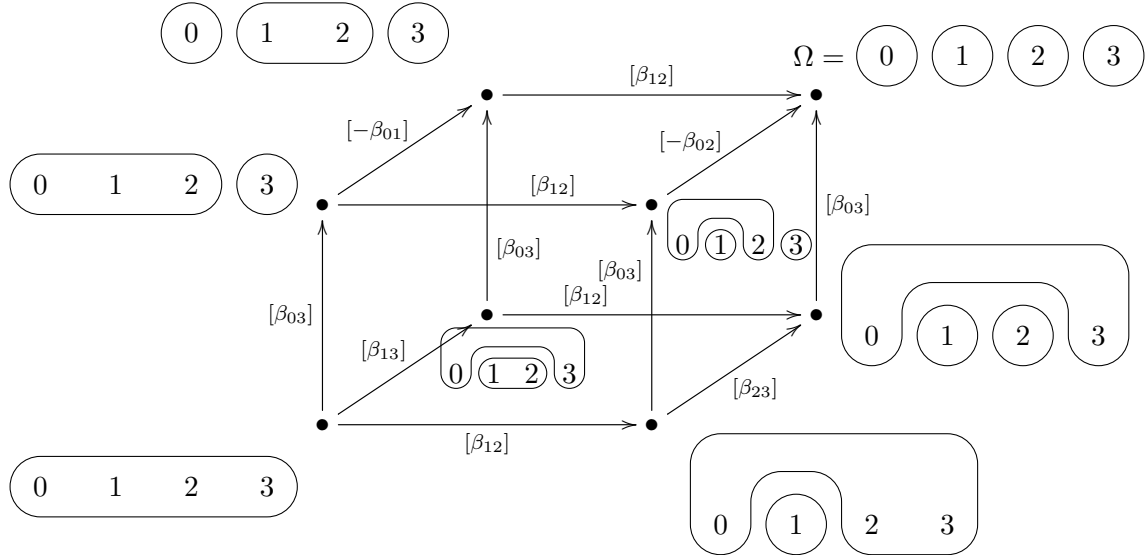
We call  $g : A \rightarrow C$  a “first factor” of  $f$  if  $g$  is irreducible. We call  $h : C \rightarrow B$  a “last factor” of  $f$  if  $h$  is irreducible.

**Definition 3.2.** A *cubical category* is a small category  $\mathcal{C}$  with the following properties.

- (1) Every morphism  $f : A \rightarrow B$  in the category has a *rank*  $rk f$  which is a nonnegative integer so that  $rk(f \circ g) = rk f + rk g$ .
- (2) If  $rk f = n$  then the factorization category of  $f$  is isomorphic to the standard  $n$ -cube category:  $Fac(f) \cong \mathcal{I}^n$ .
- (3) The forgetful functor  $Fac(f) \rightarrow \mathcal{C}$  is an embedding. In particular every morphism of rank  $n$  has  $n$  distinct first factors and  $n$  distinct last factors.
- (4) Every morphism of rank  $n$  is determined by its  $n$  first factors.
- (5) Every morphism of rank  $n$  is determined by its  $n$  last factors.

Condition (1) implies that every isomorphism has rank 0. Condition (2) implies that every rank 0 morphism is an identity map. So, a cubical category has only one object in every isomorphism class.

**Example 3.3.** This cube shows the  $3! = 6$  factorizations of the morphism of rank 3 given by the binary tree with edge vectors  $\beta_{12}, -\beta_{02}, \beta_{03}$  (given by the last three factors). In other words, the tree has an edge with positive slope from vertex 1 to vertex 2, an edge with negative slope from vertex 0 to vertex 2 and an edge with positive slope from vertex 0 to vertex 3.



In a cubical category, a collection of rank one morphisms  $f_i : X \rightarrow Y_i, i = 1, \dots, k$  are called *s-compatible* if they form the first factors of a (unique) rank  $k$  morphism  $X \rightarrow Z$ .

Similarly, a collection of rank one morphisms  $g_i : Y_i \rightarrow Z$ ,  $i = 1, \dots, k$  are called *t-compatible* if they form the last factors of a (unique) rank  $k$  morphism  $X \rightarrow Z$ .

**Proposition 3.4.** *Suppose that  $\mathcal{C}$  is a cubical category. Then the following additional properties are sufficient for the classifying space  $BC$  to be locally  $CAT(0)$  and thus a  $K(\pi, 1)$ .*

- (1) *The fundamental groupoid is a faithful functor on  $\mathcal{C}$ . I.e., two unequal morphisms  $f \neq g : X \rightarrow Y$  between any two objects give nonhomotopic paths from  $X$  to  $Y$  in  $BC$ .*
- (2) *A set of rank 1 morphisms  $f_i : X \rightarrow Y_i$  is s-compatible if and only if it is pairwise s-compatible.*
- (3) *A set of rank 1 morphisms  $g_i : Y_i \rightarrow Z$  is t-compatible if and only if it is pairwise t-compatible.*

We will prove this later after we prove that the category  $\mathcal{NP}(n)$  has these properties.

*Remark 3.5.* We have already shown that  $\mathcal{NP}(n)$  satisfies Condition (3). Indeed, when a morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  of rank, say  $k$ , is decomposed into a product of rank one morphisms, the last morphism  $\mathcal{R} \rightarrow \mathcal{Q}$  is, by definition, given by one of the  $k$  elements of  $T \subseteq E(\mathcal{Q})$ . By Theorem 1.14, such elements of  $E(\mathcal{Q})$  are compatible if and only if they are pairwise compatible.

We recall that the *fundamental groupoid* of  $\mathcal{C}$  is another category  $\pi_1\mathcal{C}$  with the same object set as  $\mathcal{C}$  but where morphisms from  $X$  to  $Y$  are defined to be homotopy classes of paths from  $X$  to  $Y$  in  $BC$ . Since any morphism  $f : X \rightarrow Y$  gives a path  $X \rightarrow Y$  whose homotopy class is a morphism  $\pi_1 f : X \rightarrow Y$  in  $\pi_1\mathcal{C}$ , there is a functor

$$\pi_1 : \mathcal{C} \rightarrow \pi_1\mathcal{C}.$$

Furthermore, this functor is universal among all functors of  $\mathcal{C}$  into all groupoids.

**3.2. Representation of  $\mathcal{NP}(n)$ .** There is a “standard representation” of the category  $\mathcal{NP}(n)$  given as follows. Let  $U_n(\mathbb{Z})$  be the group of  $n \times n$  unipotent upper triangular matrices with integer entries.

**Proposition 3.6.** *For every morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  between any two objects of  $\mathcal{NP}(n)$ , there is a matrix*

$$g[T] \in U_n(\mathbb{Z})$$

*with the following properties.*

- (1)  $g([T] \circ [S]) = g[S]g[T]$ .
- (2) *If  $[S] \neq [T] : \mathcal{P} \rightarrow \mathcal{Q}$  then  $g[T] \neq g[S]$ .*

The first property implies that  $g$  sends identity morphisms to identity matrices. The second property implies that, for  $\mathcal{P}, \mathcal{Q}$  fixed, there is at most one morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  for which  $g[T] = I_n$ .

**Corollary 3.7.** *The matrix function  $g$  induces a group homomorphism  $\pi_1\mathcal{NP}(n) \rightarrow U_n(\mathbb{Z})$  which separates morphisms and therefore show that any two distinct morphisms  $\mathcal{P} \rightarrow \mathcal{Q}$  give nonhomotopic paths in the geometric realization of  $\mathcal{NP}(n)$ .*

*Proof.* Since  $g$  is a functor from  $\mathcal{NP}(n)$  to a group which is a special case of a groupoid, it factors through the fundamental groupoid functor  $\pi_1 : \mathcal{NP}(n) \rightarrow \pi_1\mathcal{NP}(n)$ . Condition (2) implies that  $g$  separates morphisms.  $\square$



**Definition 3.8.** Recall that a morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  is given by a rooted tree  $T_\alpha$  for every part  $W_\alpha \in \mathcal{P}$ . This gives a partial ordering on the subset  $\mathcal{Q}_\alpha \subseteq \mathcal{Q}$  of all parts which lie in  $W_\alpha$ . Let  $g[T] \in U_n(\mathbb{Z})$  be given by  $g_{ij}[T] = 1$  if either  $v_i = v_j$  or all the following hold.

- (1)  $v_i, v_j$  lie in the same part of  $\mathcal{P}$ , say  $v_i, v_j \in W_\alpha$ , but in different parts  $X, Y$  of  $\mathcal{Q}_\alpha$ ,
- (2)  $i < j$  and  $X < Y$  in the order given by the tree  $T_\alpha$ .
- (3)  $j$  is minimal so that  $i < j$  and  $v_j \in Y$

and  $g_{ij}[T] = 0$  otherwise.

Since  $X, Y$  are noncrossing it follows that, if  $g_{ij}[T] = 1$  and  $i \neq j$ , then  $g_{i'j}[T] = 1$  for all other  $v_{i'} \in X$ , the part of  $\mathcal{Q}$  containing  $v_i$ .

**Example 3.9.** Let  $\mathcal{P}, \mathcal{Q}$  be the following partitions of 5:  $\mathcal{P} = \{V\}$ ,  $\mathcal{Q} = \{X, Y\}$  where  $X = \{v_2, v_3\}$ ,  $Y = \{v_1, v_4, v_5\}$ . Then there is only one morphism  $[S] : \mathcal{P} \rightarrow \mathcal{Q}$  given by the tree on  $\mathcal{Q}$  with root  $Y$ . Then

$$g[S] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

The nonzero nondiagonal entries are  $g_{24} = g_{34} = 1$  since  $j = 4$  is the smallest index of an element of  $Y$  to the right of  $X$ . Thus left multiplication by  $g[S]$  is the row operation which adds Row 4 to Row 2 and to Row 3. Let  $\Omega$  be the noncrossing partition of  $V$  into its individual elements. Let  $[T] : \mathcal{Q} \rightarrow \Omega$  be the morphism given by the binary trees on  $V_\alpha = \{2, 3\}$  and  $V_\beta = \{1, 4, 5\}$  given by  $v_2 < v_3$  and  $v_1 < v_4 < v_5$ . Then

$$g[T] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \quad g[S]g[T] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$$

The composition  $[T] \circ [S] = [T \cup \sigma_T S]$  is given by adding the edge  $-\beta_{13} = v_1 - v_3$  to  $T$ . (This is the only edge compatible with the other edges in  $T$ .) The partial ordering of vertices given by this new tree is

$$v_2 < v_3 < v_1 < v_4 < v_5$$

giving the matrix  $g[S]g[T]$  as required.

**Lemma 3.10.** *A morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  is uniquely determined by  $\mathcal{P}, \mathcal{Q}$  and the matrix  $g[T]$ .*

*Proof.* The tree  $T$  is a union of augmented binary trees on relative parallel sets. Since the augmentation of a binary tree is unique, it suffices to determine the partial ordering and thus the binary tree on every parallel set in  $\mathcal{P}$  relative to  $\mathcal{Q}$ . So, let  $\mathcal{U} = \{X_1, \dots, X_n\}$  be a parallel set in  $\mathcal{Q}_\alpha \subseteq \mathcal{Q}$ , the union of parts in  $W_\alpha \in \mathcal{P}$ . For each  $i$ , let  $v_i$  be the leftmost element of  $X_i$  and consider only the entries  $g_{ij}[T]$  corresponding to the  $n$  points  $v_i, v_j$ .

Claim: The matrix  $g[T]$  determines the root  $X_k$  of  $\mathcal{U}$ .

*Proof:* Take  $k$  maximal so that  $g_{ik} = 1$  for all  $i < k$ . For any  $j > k$  we have  $g_{jk} = 0$ . So, this holds only when  $X_k$  is the root.

By induction on  $n$ , the submatrices  $g_{ij}[T]$  for  $i, j < k$  and  $g_{ij}[T]$  for  $i, j > k$  uniquely determine the subtrees of the tree on  $\mathcal{U}$  given by deleting the root. Therefore, the matrix determines the entire binary tree on  $\mathcal{U}$  and therefore the entire rooted tree  $T$ .  $\square$

*Proof of Proposition 3.6.* We have shown Property (2). To prove Property (1), we first reduce it to the case when  $S$  has only one element. Indeed, given composable morphisms

$$\mathcal{Q} \xrightarrow{[S]} \mathcal{R} \xrightarrow{[T]} \mathcal{S}$$

where  $S$  has more than one element,  $[S]$  can be written as a composition of two shorter morphisms  $[S] = [S_1] \circ [S_2]$ . Then, by induction on  $|S|$  we have

$$g([T] \circ [S]) = g([T] \circ [S_1] \circ [S_2]) = g[S_2]g([T] \circ [S_1]) = g[S_2]g[S_1]g[T] = g[S]g[T]$$

So, it is enough to prove the base case when  $S$  has one element, say  $S = Y - X$ . There are three cases. Either  $X, Y$  are parallel with  $Y$  to the left or right of  $X$  or  $Y$  covers  $X$ .

In all cases we have  $[T] \circ [S] = [T \cup \sigma_T S]$  where  $\sigma_T S = Z - R \in E_T(\mathcal{S}, \mathcal{Q})$  where  $R$  is the root of the rooted tree  $T_X$  given by restricting  $T$  to the parts of  $\mathcal{S}$  which lie in  $X \in \mathcal{R}$  and  $Z$  is a part of  $\mathcal{S}$  which lies in  $Y$  and which depends on the case.

Case 1: Suppose  $X, Y$  are parallel with  $Y$  to the left of  $X$ .

In this case  $g[S] = I_n$  is the identity matrix. The addition of the edge  $\sigma_T S$  to  $T$  raises some parts in  $Y$  above all parts of  $X$ . However, since  $Y$  is to the left of  $X$ , this has no effect on the matrix. So,

$$g[T \cup \sigma_T S] = g[T] = g[S]g[T].$$

Case 2: Suppose that  $X, Y$  are parallel with  $Y$  to the right of  $X$ .

Let  $v_j$  be the leftmost element of  $Y$ . Then, by definition,  $g[S]$  is the identity matrix except for the  $j$ th column where  $g_{ij}[S] = 1$  if and only if either  $i = j$  or  $v_i \in X$ . The new edge  $\sigma_T S = Z - R$  goes from the root  $R$  of  $T_X$  to the leftmost part  $Z \in \mathcal{S}$  of  $Y$ . When this new edge is added, every part of  $T_X$  becomes less than  $Z$  (this makes  $g_{ij} = 1$  for all  $v_i \in X$ ) and thus, by transitivity, less than any other part which is greater than  $Z$ . In other words,  $g[T \cup \sigma_T S]$  is obtained from  $g[T]$  by adding its  $j$ th row to the rows corresponding to the elements of  $X$ . So,  $g[T \cup \sigma_T S] = g[T] = g[S]g[T]$  in this case as well.

Case 3: Suppose  $Y$  covers  $X$ .

In this case  $Z$  is the part of  $\mathcal{S}$  which contains the leftmost element  $v_j$  of  $Y$  to the right of  $X$ . As in the other cases,  $g[S]$  is equal to the identity matrix except in Column  $j$  where  $g_{ij}[S] = 1$  iff either  $i = j$  or  $v_i \in X$ . Again, the addition of  $\sigma_T S = Z - R$  makes  $Z$  greater than all parts of  $X$  in  $\mathcal{R}$  and thus changes  $g[T]$  by adding Row  $j$  to Row  $i$  for every  $v_i \in X$  which is the same as left multiplication by  $g[S]$ .

This proves Property (1) in all three cases and completes the proof of Proposition 3.6.  $\square$

**3.3.  $\mathcal{NP}(n)$  is cubical.** Let  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$  be a cluster morphism of rank  $k$ .

**Lemma 3.11.** *The factorization category  $\text{Fac}[T]$  is isomorphic to the cube  $\mathcal{I}^k$  and the forgetful functor  $\text{Fac}[T] \rightarrow \mathcal{NP}(n)$  is an embedding.*

*Proof.* Let  $T = \{E_1, \dots, E_k\}$ . Then any factorization of  $[T]$  has the form

$$\mathcal{P} \xrightarrow{[T_1]} \mathcal{R} \xrightarrow{[T_2]} \mathcal{Q}$$

where  $T_2 \subseteq T$  and  $T_1$  is uniquely determined by  $T_2$  since  $T = T_2 \cup \sigma_{T_2}(T_1)$  and  $\sigma_{T_1}$  is a bijection. Furthermore, there exists a morphism from  $\mathcal{P} \rightarrow \mathcal{R} \rightarrow \mathcal{Q}$  to

$$\mathcal{P} \xrightarrow{[T'_1]} \mathcal{R}' \xrightarrow{[T'_2]} \mathcal{Q}$$

if and only if  $T'_2 \subseteq T_2$ . And this morphism  $\mathcal{R} \rightarrow \mathcal{R}'$  is unique since it is  $\sigma_{T'_2}^{-1}$  of the complement of  $T'_2$  in  $T_2$ .

The partitions  $\mathcal{R}$  corresponding to subsets  $T_2$  of  $T$  are also unique since they are given by merging the corresponding pairs of parts of  $\mathcal{Q}$  together. Pairs not in  $T_2$  are not merged and thus lie in separate parts of  $\mathcal{R}$ . So, different  $T_2$  give different  $\mathcal{R}$ .

This description of  $Fac[T]$  proves the lemma.  $\square$

We now consider first factors of cluster morphisms. Let  $\{V\}$  be the partition of  $V = \{v_1, v_2, \dots, v_n\}$  into one part. All possible noncrossing partitions of  $V$  into two parts are given by  $\mathcal{P}_{ij} = \{X_{ij}, Y_{ij}\}$  where

$$X_{ij} = \{v_{i+1}, \dots, v_j\}, \quad 0 \leq i < j < n$$

and  $Y_{ij} = V \setminus X_{ij}$ .

**Definition 3.12.** Let  $\mathcal{C}(V)$  be the set of all morphisms of rank one:

$$[M_{ij}] : \{V\} \rightarrow \mathcal{P}_{ij}, 0 \leq i < j < n$$

given by  $M_{ij} = Y_{ij} - X_{ij}$  and

$$[\overline{P}_j] : \{V\} \rightarrow \mathcal{P}_{0j}, 0 < j < n$$

given by  $\overline{P}_j = X_{0j} - Y_{0j}$ .

*Remark 3.13.* These correspond to the objects of the cluster category of type  $A_{n-1}$  with straight orientation. Rank one morphisms are compatible if and only if the corresponding objects in the cluster category do not extend each other. Maximal pairwise compatible sets have  $n - 1$  elements and form what are called “cluster tilting objects” in the cluster category. The resulting flag complex is well-known to be a sphere of dimension  $n - 2$  with a Catalan number of simplices of maximal dimension. See [3], [4] for details.

**Theorem 3.14.** *The set  $\mathcal{C}(V)$  is the set of all morphisms from  $\{V\}$  to noncrossing partitions with two parts. The morphism  $[\overline{P}_k]$  is compatible with  $[M_{ij}]$  for  $k \notin (i, j)$  and with all other  $[\overline{P}_\ell]$ .  $[M_{ij}], [M_{k\ell}]$  are compatible if and only if they are noncrossing, i.e., the intervals  $(i, j), (k, \ell)$  are either disjoint or one contains the other. A collection of elements of the set  $\mathcal{C}(V)$  form the set of first factors of a unique cluster morphism  $\{V\} \rightarrow \mathcal{P}$  if and only if they are pairwise compatible.  $\square$*

*Remark 3.15.* The reason that morphisms in  $\mathcal{NP}(n)$  are called “cluster morphisms” is because they are given by partial clusters in the cluster category. They are defined in terms of the “c-vectors” of the cluster in this paper. The more general result in [14] is stated in the more representation theoretic terms of cluster tilting objects in the cluster categories of hereditary abelian categories called “wide subcategories” of a module category.

**Corollary 3.16.**  *$\mathcal{NP}(n)$  is a cubical category satisfying all the properties of Proposition 3.4.*

*Proof.* To prove that  $\mathcal{NP}(n)$  is cubical, it remains to prove Property (3) in the definition of cubical category. Since we have shown that the first factors are distinct, we need only show that the set of first factors determines the morphism  $[T] : \mathcal{P} \rightarrow \mathcal{Q}$ . But  $T$  is a disjoint union of rooted trees, one for each part of  $\mathcal{P}$ . So, we can deal with each part of  $\mathcal{P}$  separately and we are reduced to the theorem above.

The theorem above shows that Condition (2) in Proposition 3.4 is satisfied. The other conditions have already been verified.  $\square$

**3.4. Proof of Proposition 3.4.** Suppose that  $\mathcal{C}$  is a cubical category satisfying the conditions of Proposition 3.4. Then the classifying space of  $BC = |\mathcal{N}_\bullet \mathcal{C}|$  is a union of cubes since every sequence of composable morphisms is contained in the factorization cube of its composition. As a simplicial complex, these cubes are triangulated. However, we will ignore this subdivision of each cube and consider the structure of  $BC$  as a union of cubes.

For any object  $X$  in  $\mathcal{C}$  we define the *forward link*  $Lk_+(X)$  of  $X$  to be the simplicial complex whose vertices are all rank one morphisms with source  $X$  so that a collection of such morphisms forms a simplex if they form the first factors of some morphism in  $\mathcal{C}$ . The *backward link*  $Lk_-(X)$  of  $X$  is defined analogously.

**Lemma 3.17.** *Let  $X$  be any object of  $\mathcal{C}$ . Then the link of the vertex  $X$  in  $BC$  is isomorphic as a simplicial complex to the join of the forward link of  $X$  and its backward link. Consequently, it is a flag complex.*  $\square$

It remains to show that the cubes in  $BC$  intersect in faces. We were not able to prove this. However, if we cut each  $k$ -cube into  $2^k$  smaller cubes of half size, the smaller cubes will satisfy this property and still have good links. If  $C$  is one of these small cubes in  $BC$ , we use the notation  $\bar{C}$  to denote the original ‘‘big cube’’ of  $BC$  of which  $C$  is one piece. Note that each vertex of a small cube is the center point of some big cube. Also, each vertex of a big cube lies in exactly one of its small cubes.

By definition, the classifying space  $BC$  is a union of simplices  $\Delta^k$  indexed by sequences of  $k$  composable morphisms  $f_1, \dots, f_k$ . By definition of a cubical category, these morphisms go through  $k + 1$  distinct objects of  $\mathcal{C}$ . And each point  $x \in BC$  lies in the interior of a unique simplex. Let  $\varphi(x)$  denote the composition of the morphisms defining this simplex. Then the interior points of the factorization cube of a morphism  $f$  are exactly those point  $x$  with  $\varphi(x) = f$ .

**Lemma 3.18.** *Suppose that  $C_1, C_2$  are small cubes whose intersection  $C_1 \cap C_2$  contains a point in the interior of each small cube. Then  $C_1 = C_2$ .*

*Proof.* Let  $\bar{C}_1, \bar{C}_2$  be factorization cubes of  $f_1, f_2$  respectively. If  $C_1 \cap C_2$  contains a point interior to both then  $f_1 = f_2$  and  $\bar{C}_1 = \bar{C}_2$ . This implies  $C_1 = C_2$  since distinct small cubes in a big cube meet only on their faces.  $\square$

**Lemma 3.19.** *Given any two small cubes  $C_1, C_2$ , there is a big cube  $C_0$  containing  $C_1 \cap C_2$  with the property that  $C_i \cap C_0$  is a small cube face of  $C_i$  for  $i = 1, 2$ .*

Suppose for a moment that this is true.

**Proposition 3.20.** *For any two small cubes  $C_1, C_2$  in  $BC$ , the intersection  $C_1 \cap C_2$  is a common face.*

*Proof.* Take the large cube  $C_0$  given by the lemma. Then

$$C_1 \cap C_2 = (C_1 \cap C_0) \cap (C_2 \cap C_0)$$

Since each  $C_i \cap C_0$  is a small cube in the standard big cube  $C_0$ , they intersect correctly.  $\square$

*Proof of Lemma 3.19.* Each big cube  $\overline{C}_i$  is the factorization cube of a morphism  $f_i : X_i \rightarrow Y_i$  and  $C_i$  contains exactly one object  $Z_i$  in one factorization  $X_i \rightarrow Z_i \rightarrow Y_i$  of  $f_i$ . The edges of the cube  $C_i$  adjacent to vertex  $Z_i$  are the first factors of  $h_i$  and the last factors of  $g_i$  and any face of  $\overline{C}_i$  which contains  $Z_i$  is uniquely determined by the subset of the first factors of  $h_i$  and last factors of  $g_i$  which are contained in that face.

Let  $x \in C_1 \cap C_2$ . Then  $x$  lies in the interior of the cube  $C_x = \text{Fac}(\varphi(x))$  which must be a face of both  $\overline{C}_1, \overline{C}_2$ . So,  $Z_1, Z_2$  are vertices of  $C_x$ . In particular,  $Z_2$  is a vertex of  $\overline{C}_1$ . So,  $f_1 : X_1 \rightarrow Y_1$  factors through  $Z_2$ . Since  $\mathcal{C}$  is cubical, this factorization is unique  $f_1 : X_1 \xrightarrow{g_0} Z_2 \xrightarrow{h_0} Y_1$ .

Let  $S$  be the set of all last factors of  $g_0 : X_1 \rightarrow Z_2$  which are also last factors of  $g_2 : X_2 \rightarrow Z_2$ . Then  $S$  is the set of last factors of a unique morphism  $g : X_0 \rightarrow Z_2$  which factors both  $g_0$  and  $g_2$ . Similarly, let  $T$  be the set of all first factors of  $h_0 : Z_2 \rightarrow Y_1$  which are also first factors of  $h_2 : Z_2 \rightarrow Y_2$ . Then  $T$  is the set of first factors of a unique morphism  $h : Z_2 \rightarrow Y_0$ .

The factorization cube  $C_0$  of  $h \circ g : X_0 \rightarrow Y_0$  is a face of both  $\overline{C}_1$  and  $\overline{C}_2$  and contains  $C_x$  for all  $x \in C_1 \cap C_2$ . Therefore,  $C_0$  contains  $C_1 \cap C_2$ . Since  $C_0$  is a face of each  $\overline{C}_i$ , the intersection  $C_0 \cap C_i$  is a face of  $C_i$ .  $\square$

**Lemma 3.21.** *Let  $v$  be a vertex of a small cube  $C$ . Then  $v$  is the center point of the factorization cube  $C_0$  of  $f = \varphi(v) : X \rightarrow Y$  of dimension, say  $k$ , and the link of  $v$  in  $BC$  is isomorphic to the joint of the backward link of  $X$  with the forward link of  $Y$  and  $k$  zero spheres:*

$$\text{Lk}(v) \cong \text{Lk}_-(X) * \text{Lk}_+(Y) * S^0 * \cdots * S^0$$

*In particular the link of  $v$  is a flag complex.*

*Proof.* The big cube  $\overline{C}$  is the factorization cube of a morphism which factors uniquely through  $f$ , say as

$$A \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} B$$

Such factorizations are uniquely determined by the last factors of  $g$  and the first factors of  $h$ , i.e., by a simplex in  $\text{Lk}_-(X)$  and another simplex in  $\text{Lk}_+(Y)$ . So,  $\text{Lk}_-(X) * \text{Lk}_+(Y)$  is the link of the cube  $C_0$  in  $BC$ . Since the cube  $C_0$  is cut up into  $2^k$  little cubes, the link of the vertex  $v$  in  $C_0$  is the join of  $k$  copies of  $S^0$ . And the link of  $v$  in  $BC$  is the join of these two links as claimed.  $\square$

This completes the proof of the general Proposition 3.4. Since  $\mathcal{NP}(n)$  has been shown to satisfy the hypotheses of the proposition we conclude the theorem.

**Theorem 3.22.**  *$BN\mathcal{P}(n)$  is locally  $CAT(0)$  and thus a  $K(\pi, 1)$ .*

#### 4. FUNDAMENTAL GROUP

In this section we will compute the fundamental group of  $BN\mathcal{P}(n)$ .

**Theorem 4.1.** *The fundamental group of  $BN\mathcal{P}(n)$  is the group  $G(A_{n-1})$  having the following presentation. The generators are  $x_{ij}$  where  $1 \leq i < j \leq n$  with relations:*

- (1)  $[x_{ij}, x_{jk}] = x_{ik}$  for all  $1 \leq i < j < k \leq n$ .
- (2)  $[x_{ij}, x_{k\ell}] = 1$  if  $x_{ij}, x_{k\ell}$  are noncrossing in the sense that the closed intervals  $[i, j], [k, \ell]$  are either disjoint or one is contained in the interior of the other.

Here  $[x, y] := y^{-1}xyx^{-1}$ .

**4.1. Generators.** For the basepoint of  $BN\mathcal{P}(n)$  we take the partition  $\Omega$  of  $V = \{v_1, \dots, v_n\}$  into one point sets. Let  $\mathcal{P}_{ij}$  be the partition of giving by merging  $v_i, v_j, i < j$ , into one part  $\{v_i, v_j\}$ . Then we have two morphisms  $[\beta_{ij}], [-\beta_{ij}] : \mathcal{P}_{ij} \rightarrow \Omega$  given by  $\beta_{ij} = v_j - v_i$  and  $-\beta_{ij} = v_i - v_j$ .

Then  $x_{ij}$  is defined to be the homotopy class of the loop  $[-\beta_{ij}]^{-1}[\beta_{ij}]$  at  $\Omega$  where we use the convention that paths are always composed left to right. To show that the  $x_{ij}$  generate  $\pi_1 BN\mathcal{P}(n)$ , we use the following equivalence relation on morphisms.

Given two morphisms  $[T], [S] : \mathcal{S} \rightarrow \Omega$ , we say that  $[T] \sim [S]$  if there is a sequence of morphisms  $[T] = [T_0], [T_1], \dots, [T_m] = [S]$  so that  $[T_i], [T_{i+1}]$  share a common first factor  $\mathcal{S} \rightarrow \mathcal{R}_i$  for each  $i$ .

**Lemma 4.2.** *Let  $\mathcal{S}$  be a noncrossing partition of rank  $k \geq 2$  then any two morphisms  $[T], [S] : \mathcal{S} \rightarrow \Omega$  are equivalent.*

*Proof.* Suppose first that  $\mathcal{S}$  has at least two parts  $X, Y$  with more than one element. Then  $[T], [S]$  are given by binary trees on each of these parts. Let  $[R]$  be any morphism which is equal to  $[T]$  on  $X$  and  $[S]$  on  $Y$ . Then the part of  $[T]$  on  $X$  gives a common first factor for  $[T], [R]$ . So  $[T] \sim [R]$ . Similarly,  $[R] \sim [S]$ . So,  $[T] \sim [S]$ .

Now suppose that  $\mathcal{S}$  has only one part  $W$  which is not a singleton. Then  $W$  has  $k + 1$  elements and the forward link of  $\mathcal{S}$  is a triangulation of the sphere  $S^{k-1}$  into a Catalan number of simplices. When  $k \geq 2$  this is connected and thus any two morphisms are equivalent.  $\square$

**Proposition 4.3.** *The fundamental group of  $BN\mathcal{P}(n)$  is generated by the loops  $x_{ij}$ .*

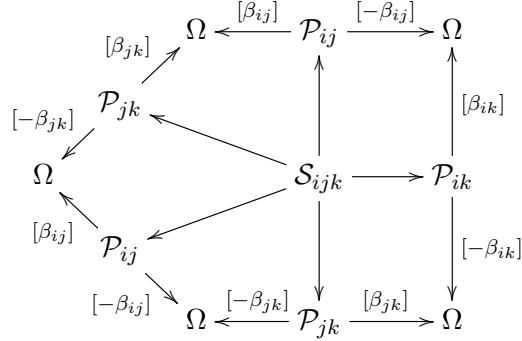
*Proof.* Choosing a morphism from every object to  $\Omega$  we see that every loop at  $\Omega$  is a composition of loops of the form  $[T]^{-1}[S]$  where  $[T], [S]$  are morphisms  $\mathcal{S} \rightarrow \Omega$ . If  $\mathcal{S}$  has rank 1 then this loop is  $x_{ij}$  or its inverse for some  $i, j$ . Otherwise,  $[S] \sim [T]$  by the lemma which implies that  $[T]^{-1}[S]$  is a composition of loops going through some object  $\mathcal{R}$  of smaller rank than  $\mathcal{S}$ . The proposition follows by induction on that rank.  $\square$

**4.2. Relations.** Let  $(i, j), (k, \ell)$  be two noncrossing pairs of numbers between 1 and  $n$ . Then, there is a noncrossing partition  $\mathcal{S}$  of rank 2 having  $\{v_i, v_j\}$  and  $\{v_k, v_\ell\}$  as two of its parts. There are exactly four morphisms  $\mathcal{S} \rightarrow \Omega$  giving four factorization squares (2-cubes):

$$\begin{array}{ccccc}
 \Omega & \xleftarrow{[-\beta_{ij}]} & \mathcal{P}_{ij} & \xrightarrow{[\beta_{ij}]} & \Omega \\
 \uparrow [\beta_{k\ell}] & & \uparrow & & \uparrow [\beta_{k\ell}] \\
 \mathcal{P}_{k\ell} & \xleftarrow{\quad} & \mathcal{S} & \xrightarrow{\quad} & \mathcal{P}_{k\ell} \\
 \downarrow [-\beta_{k\ell}] & & \downarrow & & \downarrow [-\beta_{k\ell}] \\
 \Omega & \xleftarrow{[-\beta_{ij}]} & \mathcal{P}_{ij} & \xrightarrow{[\beta_{ij}]} & \Omega
 \end{array}$$

Together, these four squares give a homotopy  $x_{ij}x_{k\ell} \simeq x_{k\ell}x_{ij}$  giving Relation (2) in the theorem.

To obtain the other relation consider the partition  $\mathcal{S}_{ijk}$  of rank 2 having one part  $\{v_i, v_j, v_k\}$  and the other parts all singletons. Then there are exactly five morphisms  $\mathcal{S}_{ijk} \rightarrow \Omega$  giving five factorization squares in  $BN(n)$  which fit into a pentagon as follows.



Starting from the lower left  $\Omega$ , the two paths going up to the upper left  $\Omega$  are  $x_{ij}x_{jk} \simeq x_{jk}x_{ik}x_{ij}$  giving Relation (1).

We now need to show that there are no other relations. We use the fact that, for any CW complex with one 0-cell, the generators of  $\pi_1$  are given by the 1-cells and the relations are given by the 2-cells.

**Theorem 4.4.** *The classifying space  $BN\mathcal{P}(n)$  is an  $n-1$  dimensional CW-complex having one cell  $e(\mathcal{S})$  of dimension  $k$  for every noncrossing partition  $\mathcal{S}$  of rank  $k$ . The  $k$ -cell  $e(\mathcal{S})$  is the union of all factorization cubes of all morphisms  $\mathcal{S} \rightarrow \Omega$ .*

Assuming the theorem, the 1-cells are all the loops

$$\Omega \xleftarrow{[-\beta_{ij}]} \mathcal{P}_{ij} \xrightarrow{[\beta_{ij}]} \Omega$$

The 2-cells are the squares and pentagons given above giving Relations (2), (1). The higher cells do not affect the fundamental group.

*Proof.* Given any noncrossing partition  $\mathcal{S} = \{X_1, \dots, X_m\}$ , a morphism  $[T] : \mathcal{S} \rightarrow \Omega$  is a product of morphism  $[T_i]$  one for each part  $X_i$  of  $\mathcal{S}$ . The forward link  $Lk_+(\mathcal{S})$  is a join of spheres

$$Lk_+(\mathcal{S}) = *S^{k_i-2}$$

where  $k_i = |X_i|$ . The union of factorization cubes for the morphisms  $\mathcal{S} \rightarrow \Omega$  is a product of disks of dimension  $k_i - 1$ . The key point is to show that these cells meet lower dimensional cells only along their boundaries. However, this is easy. The interior of the cell  $e(\mathcal{S})$  is the set of all points  $x \in BN\mathcal{P}(n)$  with the property that  $\varphi(x)$  is a morphism with source  $\mathcal{S}$ . The boundary of the cell consists of unions of those faces of factorization cubes of morphisms  $\mathcal{S} \rightarrow \Omega$  which do not include  $\mathcal{S}$  as vertex. These are morphisms having source  $\mathcal{R}$  of smaller rank than  $\mathcal{S}$  which occur in factorizations  $\mathcal{S} \rightarrow \mathcal{R} \rightarrow \Omega$ . Therefore,  $BN\mathcal{P}(n)$  is a CW complex as claimed.  $\square$

## 5. RELATION TO CLUSTER CATEGORIES

The purpose of this section is to explain the relationship between the category of noncrossing partitions as explained in detail in this paper and the category of [8] which we denote  $HK$ . Details go beyond the scope of this paper and will be explained in another paper. The basic topological difference is that the category  $HK$  has an initial object 0 and is therefore contractible. The same for the opposite category  $HK^{op}$  where 0 is the terminal object. The category  $\mathcal{NP}(n)$  has a Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$  of morphisms from  $\{1, \dots, n\}$  to the object  $\Omega$  which corresponds to 0. There does not appear to be a

functor from one category to the other. However, there is a third category that is related to both. This is an extended version of the “cluster morphism category” from [14].

**5.1. Preliminaries on categories.** Suppose that  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor and let  $X \in \mathcal{D}$ . Then the *comma category*  $F \downarrow X$  ([19]) is defined to be the category of pairs  $(Y, f)$  where  $Y$  is an object in  $\mathcal{C}$  and  $f : FY \rightarrow X$  is a morphism in  $\mathcal{D}$ . A morphism  $(Y, f) \rightarrow (Z, g)$  is a morphism  $h : Y \rightarrow Z$  in  $\mathcal{C}$  so that  $f = g \circ Fh : FY \rightarrow FZ \rightarrow X$ . In the special case when  $F$  is the identity functor  $\mathcal{C} \rightarrow \mathcal{C}$ , we use the notation  $\mathcal{C} \downarrow X$  for  $id_{\mathcal{C}} \downarrow X$ . Dually, let  $X \downarrow F$  denote the (*left*) *comma category* with objects all pairs  $(f, Y)$  where  $Y \in \mathcal{C}$  and  $f : X \rightarrow FY$ . A morphism  $(f, Y) \rightarrow (g, Z)$  in  $X \downarrow F$  is a morphism  $h : Y \rightarrow Z$  in  $\mathcal{C}$  so that  $g = Fh \circ f : X \rightarrow FY \rightarrow FZ$ . The following is an easy exercise.

**Proposition 5.1.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}'$  be a functor from  $\mathcal{C}$  to a subcategory  $\mathcal{D}'$  of  $\mathcal{D}$  and let  $J : \mathcal{D}' \hookrightarrow \mathcal{D}$  be the inclusion functor. Let  $X \in \mathcal{D}'$ . Then  $F \downarrow X$  is a full subcategory of  $JF \downarrow X$ .*

Given two equivalent Krull-Schmidt categories  $\mathcal{C}, \mathcal{D}$  we define an *functorial bijection*  $f : \text{Ind } \mathcal{C} \rightarrow \text{Ind } \mathcal{D}$  to be a bijection between the sets of isomorphism classes of indecomposable objects of  $\mathcal{C}, \mathcal{D}$  which comes from an equivalence of categories  $\mathcal{C} \cong \mathcal{D}$ . Thus, for example, if  $\mathcal{C}, \mathcal{D}$  have only finitely many indecomposable objects, then there are at most finitely many functorial bijections  $\text{Ind } \mathcal{C} \rightarrow \text{Ind } \mathcal{D}$  although there may be infinitely many equivalences.

**5.2. Cluster morphism category.** Let  $\mathcal{A}, \mathcal{B}$  be hereditary abelian categories which are equivalent to module categories of finite dimensional algebras over the same field, say  $\mathbb{k}$ . Let  $\mathcal{C}_{\mathcal{A}}, \mathcal{C}_{\mathcal{B}}$  be the corresponding cluster categories. (See [3] for details.) A *cluster morphism*  $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{C}_{\mathcal{B}}$  is defined to be a pair  $([T], f)$  where

- (1)  $[T]$  is (the isomorphism class of) a partial cluster tilting object  $T \in \mathcal{C}_{\mathcal{A}}$  and
- (2)  $f : \text{Ind}(T^{\perp}) \rightarrow \text{Ind } \mathcal{B}$  is a functorial bijection.

Recall that  $T^{\perp}$  is the full subcategory of the category  $\mathcal{A}$  of all modules  $M$  so that  $\text{Hom}(|T|, M) = 0 = \text{Ext}(|T|, M)$  where  $|T|$  is the underlying module of  $T$  (replacing shifted projective summands  $P[1]$  with  $|P[1]| = P$ ). Composition of morphisms uses the bijection between ordered clusters and signed exceptional sequences. See [14]. As a special case,  $\text{Hom}(\mathcal{C}_{\mathcal{A}}, 0)$  is the set of isomorphism classes of cluster-tilting objects of  $\mathcal{C}_{\mathcal{A}}$ .

Let  $\mathcal{G}_{\mathbb{k}}$  denote the category whose objects are hereditary abelian  $\mathbb{k}$ -categories as discussed above and whose morphisms  $\mathcal{A} \rightarrow \mathcal{B}$  are defined to be the cluster morphism  $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{C}_{\mathcal{B}}$  as defined above.

In the paper [14] we considers the following related category. For a fixed object  $\mathcal{A} \in \mathcal{G}_{\mathbb{k}}$ , let  $\mathcal{G}(\mathcal{A})$  denote the category of all finitely generated wide subcategories  $\mathcal{W} \subseteq \mathcal{A}$ . These are exactly the categories which occur as perpendicular categories of partial cluster tilting objects  $T$  in  $\mathcal{C}_{\mathcal{A}}$ . A morphism  $\mathcal{W}_1 \rightarrow \mathcal{W}_2$  in  $\mathcal{G}(\mathcal{A})$  is defined to be an isomorphism class  $[S]$  of a partial cluster tilting object  $S$  in  $\mathcal{W}_1$  so that  $S^{\perp} \cap \mathcal{W}_1 = \mathcal{W}_2$ . There is a functor  $J : \mathcal{G}(\mathcal{A}) \rightarrow \mathcal{G}_{\mathbb{k}}$  given by the identity on objects  $J\mathcal{W} = \mathcal{W}$  and on a morphism  $[S] : \mathcal{W}_1 \rightarrow \mathcal{W}_2$  we take  $J[S]$  to be the pair  $([S], f)$  where  $f : \text{Ind}(S^{\perp} \cap \mathcal{W}_1) \rightarrow \text{Ind } \mathcal{W}_2$  is the identity mapping.

**Proposition 5.2.** *Let  $\overline{\mathcal{G}}_{\mathbb{k}}$  denote the quotient category of  $\mathcal{G}_{\mathbb{k}}$  with the same objects but with equivalence classes of morphisms where  $([T], f) \sim ([S], g)$  if  $T^{\perp} = S^{\perp}$  and  $f = g$ . Let  $F : \mathcal{G}_{\mathbb{k}} \rightarrow \overline{\mathcal{G}}_{\mathbb{k}}$  be the forgetful functor. Then  $\mathcal{G}(\mathcal{A})$  is equivalent to the comma category  $\mathcal{A} \downarrow F$ .*

*Proof.* An object of  $\mathcal{A} \downarrow F$  is a pair  $(f, \mathcal{B})$  where  $f : \text{Ind } \mathcal{W}_0 \cong \text{Ind } \mathcal{B}$  is a functorial bijection where  $\mathcal{W}_0$  is a finitely generated wide subcategory of  $\mathcal{A}$ . If  $(g, \mathcal{C})$  is another object of



$\mathcal{A} \downarrow F$  we have  $g : \text{Ind } \mathcal{W}_1 \cong \text{Ind } \mathcal{C}$ . A morphism  $(f, \mathcal{B}) \rightarrow (g, \mathcal{C})$  is a cluster morphism  $([T], h) : \mathcal{B} \rightarrow \mathcal{C}$  so that the following diagram commutes.

$$\begin{array}{ccccc}
\text{Ind } \mathcal{W}_0 & \xrightarrow{f} & \text{Ind } \mathcal{B} & & \\
\subseteq \uparrow & & \subseteq \uparrow & & \\
\text{Ind } \mathcal{W}_1 & \longrightarrow & \text{Ind}(T^\perp \cap \mathcal{B}) & \xrightarrow{h} & \text{Ind } \mathcal{C} \\
& & \searrow & \nearrow & \\
& & & g & 
\end{array}$$

In other words,  $\mathcal{W}_1 = \mathcal{W}_0 \cap S^\perp$  where  $S = f^{-1}(T)$ . An equivalence of categories  $\mathcal{A} \downarrow F \cong \mathcal{G}(\mathcal{A})$  is given by sending  $(f, \mathcal{B})$  to  $\mathcal{W}_0$  and  $([T], h) : (f, \mathcal{B}) \rightarrow (g, \mathcal{C})$  to  $[f^{-1}(T)] : \mathcal{W}_0 \rightarrow \mathcal{W}_1$ . The inverse  $\mathcal{G}(\mathcal{A}) \cong \mathcal{A} \downarrow F$  is given by taking  $\mathcal{W}$  to  $(id, \mathcal{W})$ .  $\square$

Since  $\mathcal{NP}(n)$  is equivalent to the cluster morphism category of the path algebra of  $A_{n-1}$  with straight orientation we conclude the following (for any field  $\mathbb{k}$ ).

**Corollary 5.3.** *The category  $\mathcal{NP}(n+1)$  is equivalent to the comma category  $\mathbb{k}A_n \downarrow F$  where  $F : \mathcal{G}_{\mathbb{k}} \rightarrow \overline{\mathcal{G}}_{\mathbb{k}}$  is the forgetful functor and  $A_n$  denotes the quiver  $1 \leftarrow 2 \leftarrow \dots \leftarrow n$ .*

**5.3. The category of Hubery-Krause.** We recall that the objects of the Hubery-Krause category  $HK$  are pairs  $(L, E)$  where  $L \cong \mathbb{Z}^n$  (for  $n$  variable) with an Euler form  $\langle \cdot, \cdot \rangle$  coming from some finite dimensional hereditary algebra  $\Lambda$  over a field  $\mathbb{k}$  and  $E \subset L$  is the set of dimension vectors of a complete *exceptional sequence up to sign*. (All  $2^n$  possible signs for an exceptional sequence are allowed in [8]. But in [14] not all signs are allowed. E.g., there are only  $n!C_{n+1}$  *signed exceptional sequences* of type  $A_n$  corresponding to the  $n!$  permutations of the  $C_{n+1}$  clusters.) A morphism  $(L', E') \rightarrow (L, E)$  is a linear embedding  $\varphi : L' \hookrightarrow L$  which is an isometry with respect to the Euler forms and which sends  $E'$  to the set of dimension vectors of an exceptional sequence up to sign. The representation theoretic language can be removed from the definition since the dimension vectors of exceptional sequences in  $L$  depends only on  $E$  and the Euler form.

There is a (contravariant) forgetful functor  $\mathcal{G}_{\mathbb{k}} \rightarrow HK^{op}$  given by sending  $\mathcal{A}$  to the pair  $(K_0(\mathcal{A}), E)$  where  $E = \{[S_i]\}$  is the basis of  $K_0(\mathcal{A})$  given by the simple modules. (See [8], Lemma 5.3.) A morphism  $([T], f) : \mathcal{A} \rightarrow \mathcal{B}$  is sent to the monomorphism  $f^* : K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A})$  induced by  $f$ . We denote this functor by  $K_0$ . Thus  $K_0(\mathcal{A}) = (K_0(\mathcal{A}), E)$  with  $E$  understood. Since  $f$  is uniquely determined by  $f^*$ , we get the following.

**Lemma 5.4.** *The functor  $K_0 : \mathcal{G}_{\mathbb{k}} \rightarrow HK^{op}$  factors through the forgetful functor  $F : \overline{\mathcal{G}}_{\mathbb{k}} \rightarrow HK^{op}$  and the induced functor  $\overline{\mathcal{G}}_{\mathbb{k}} \rightarrow HK^{op}$  is faithful (but not full).*  $\square$

An example of a morphism in  $HK^{op}$  which is not in  $\overline{\mathcal{G}}_{\mathbb{k}}$  is multiplication by  $-1$  (corresponding to the shift [1] in the derived category). By Proposition 5.1 we get the following.

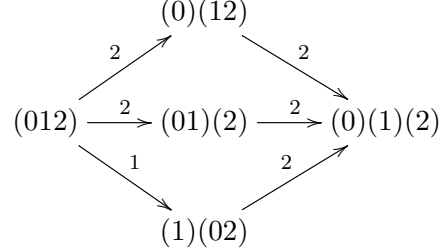
**Theorem 5.5.** *Let  $\mathcal{A}$  be an object of  $\mathcal{G}_{\mathbb{k}}$ . Then the opposite category of  $\mathcal{G}(\mathcal{A})$  is equivalent to a full subcategory of  $K_0 \downarrow K_0(\mathcal{A})$ .*  $\square$

By Corollary 5.3, we get the following special case.

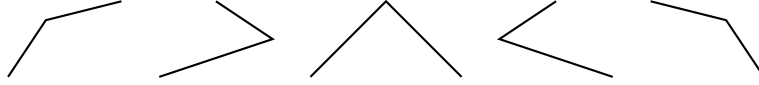
**Corollary 5.6.** *The opposite category of  $\mathcal{NP}(n+1)$  is equivalent to a full subcategory of  $K_0 \downarrow K_0(\mathbb{k}A_n)$ .*

**Example 5.7.** Let  $\mathcal{A}_2$  be the module category of  $\mathbb{k}A_2$  the path algebra of the quiver  $1 \leftarrow 2$ . Then  $\mathcal{A}_2$  has five wide subcategories:  $0, \mathcal{A}(S_1), \mathcal{A}(S_2), \mathcal{A}(P_2), \mathcal{A}_2$ . These correspond to the

five noncrossing partitions of the set  $\{0, 1, 2\}$ :  $(0)(1)(2)$ ,  $(01)(2)$ ,  $(0)(12)$ ,  $(1)(02)$ ,  $(012)$ , resp. Up to isomorphism, these are the five objects of  $\mathcal{NP}(3)$  and there are 11 morphisms of rank one and 5 morphisms of rank two between these arranged as follows.



The labels on the arrows indicate the number of rank one morphisms. There are five rank 2 morphisms given by the binary trees:



Each rank 2 morphism has  $2! = 2$  possible factorizations given by the possible orderings of the set of edges. These give the  $2!5 = 10$  signed exceptional sequences. There are 12 exceptional sequences up to sign. The two extraneous ones are:  $(\pm\beta, -\alpha_2)$ . These are not signed exceptional sequences since  $\alpha_2$  is not projective. (The only negative objects in the cluster category are the shifted projective objects.)

In the category  $HK$ , we have  $K_0(\mathbb{k}A_2) = (\mathbb{Z}^2, \{\alpha_1, \alpha_2\})$  with Euler form given by the Euler matrix  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  and  $\alpha_i = e_i$  are the basis vectors. There are six morphisms  $K_0(\mathbb{k}) = (\mathbb{Z}, \alpha_1) \rightarrow K_0(\mathbb{k}A_2)$  given by sending  $\alpha_1$  to the six roots  $\pm\alpha_1, \pm\alpha_2, \pm\beta$  where  $\beta = \alpha_1 + \alpha_2$ . Since  $K_0(\mathbb{k}A_2)$  has six automorphisms, the comma category  $K_0 \downarrow K_0(\mathbb{k}A_2)$  has 13 objects (up to isomorphism):

- (1) one object  $(0, 0)$  corresponding to the unique morphism  $(0, \emptyset) \hookrightarrow K_0(\mathbb{k}A_2)$ ,
- (2) six objects  $(\mathbb{k}A_1, (\beta_i)_*)$  where  $\beta_i$  runs through the six roots of the root system,  $(\beta_i)_* : K_0(\mathbb{k}A_1) \rightarrow K_0(\mathbb{k}A_2)$  being the isometric embedding sending  $\alpha_1$  to  $\beta_i$ ,
- (3) six objects  $(\mathbb{k}A_2, \phi^i)$ ,  $i = 0, \dots, 5$  where  $\phi \in \text{Aut}(K_0(\mathbb{k}A_2)) = \mathbb{Z}/6$  is a generator.

The embedding of  $\mathcal{NP}(3)$  as a full subcategory of  $K_0 \downarrow K_0(\mathbb{k}A_2)$  sends the five objects of  $\mathcal{NP}(3)$  to:

NCP	wide subcategory of $\mathcal{A}_2$	object of $K_0 \downarrow K_0(\mathbb{k}A_2)$	$T$
$(0)(1)(2)$	$0$	$(0, 0)$	$T = T_1 \oplus T_2$ (5 cases)
$(0)(12)$	$\mathcal{A}(S_2)$	$(\mathbb{k}A_1, (\alpha_2)_*)$	$T = P_1, P_1[1]$
$(01)(2)$	$\mathcal{A}(S_1)$	$(\mathbb{k}A_1, (\alpha_1)_*)$	$T = P_2, P_2[1]$
$(1)(02)$	$\mathcal{A}(P_2)$	$(\mathbb{k}A_1, (\beta)_*)$	$T = S_2$
$(012)$	$\mathcal{A}_2$	$(\mathbb{k}A_2, \phi^0 = id)$	$T = 0$

The last column lists the morphisms from  $\mathcal{A}_2$  to the object in terms of representation theory. For example, there is only one morphism  $(012) \rightarrow (1)(02)$  given by the cluster  $T = S_2$ . Since  $S_2$  is not projective, there is no  $S_2[1]$ . In terms of the noncrossing partition,  $(02)$  covers  $(1)$ , so the ordering is determined.

The category  $K_0 \downarrow K_0(\mathbb{k}A_2)$  has  $6 \times 7 = 42$  rank one morphisms. Of these, 11 lie in  $\mathcal{NP}(3) \cong \mathcal{G}(\mathcal{A}_2)$ . For each  $i$  there are two morphisms  $0 \rightarrow (\mathbb{k}A_1, \beta_i)$  (of these the six that

lie in  $\mathcal{G}(\mathcal{A}_2)$  are the ones corresponding to the three positive roots) and five morphisms  $(\mathbb{k}A_1, \beta_*) \rightarrow (\mathbb{k}A_2, \phi^i)$ . Only the five where  $i = 0, \phi^0 = id$  occurs in  $\mathcal{G}(\mathcal{A}_2)$ .

The fundamental groups of  $\mathcal{NP}(3)$  and  $K_0 \downarrow K_0(\mathbb{k}A_2)$  are:

$$\pi_1(\mathcal{NP}(3)) = \langle x_1, x_2, x_3 : x_2 = [x_1, x_3] \rangle = F_2$$

where  $[x, y] := y^{-1}xyx^{-1}$  and

$$\pi_1(K_0 \downarrow K_0(\mathbb{k}A_2)) = \langle x_1, \dots, x_6 : x_i = [x_{i-1}, x_{i+1}], i = 1, \dots, 6 \rangle$$

The embedding of  $\mathcal{NP}(3)$  into  $K_0 \downarrow K_0(\mathbb{k}A_2)$  sends  $x_i$  to  $x_i$  for  $i = 1, 2, 3$ .

The relation between the categories of noncrossing partitions given in this paper and the one in [8] is apparently very complicated even in the smallest examples. However, the following diagram summarizes the connection.

$$\mathcal{NP} \hookrightarrow \mathcal{G}_{\mathbb{k}} \rightarrow \overline{\mathcal{G}}_{\mathbb{k}} \hookrightarrow HK^{op}$$

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