CONTINUOUS FROBENIUS CATEGORIES

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Abstract. We introduce continuous Frobenius categories. These are topological categories which are constructed using representations of the circle over a discrete valuation ring. We show that they are Krull-Schmidt with one indecomposable object for each pair of (not necessarily distinct) points on the circle. By putting restrictions on these points we obtain various Frobenius subcategories. The main purpose of these Frobenius categories is to give a precise and elementary description of the triangulated structure of their stable categories, which are cluster categories for certain values of the parameters. These *continuous cluster categories* are being developed in concurrently written papers.

The standard construction of a cluster category of a hereditary algebras is to take the orbit category of the derived category of bounded complexes of finitely generated modules over the algebra:

$$\mathcal{C}_H \cong \mathcal{D}^b(mod\, H)/F$$

where F is a triangulated autoequivalence of $\mathcal{D}^b(mod\, H)$ [2]. In this paper we construct continuous versions of the cluster categories of type A_n. These *continuous cluster categories* are "continuously triangulated" categories having uncountably many indecomposable objects and containing the finite and countable cluster categories of type A_n and A_∞ as subquotients. Cluster categories of type A_n and A_∞ were also studied in [3], [6], [12].

The reason for the term *continuous* in the names of the categories is the fact that the categories that we define and consider in this paper are topological categories with continuous structure maps (Section 0). The continuity requirement implies that there are two possible topologically inequivalent triangulations of the continuous cluster category given by the two 2-fold covering spaces of the Moebius band: connected and disconnected. We consider both cases (Remarks 3.1.8, 3.4.2).

The term *cluster* in the names of the categories is justified in [8] where it is show that the category $\mathcal{C}_\pi$ has a cluster structure where cluster mutation is given using the triangulated structure (see [1]) and that the categories $\mathcal{C}_c$ also have a cluster structure for specific values of c. For the categories $\mathcal{C}_\phi$, we have partial results ($\mathcal{C}_\phi$ has an m-cluster structure in certain cases). This paper is the first in a series of papers. The main purpose of this paper is to give a concrete and self-contained description of the triangulated structures of these continuous cluster categories being developed in concurrently written papers [8, 9].

We will use representations of the circle over a discrete valuation ring R to construct continuous Frobenius R-categories $\mathcal{F}_\pi$, $\mathcal{F}_c$ and $\mathcal{F}_\phi$ whose stable categories (triangulated by [4]) are isomorphic to the continuous categories $\mathcal{C}_\pi$, $\mathcal{C}_c$ and $\mathcal{C}_\phi$, thus inducing continuous triangulated structure on these topological K-categories ($K = R/\mathfrak{m}$).

In Section 1. we define representations of the circle; a representation of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ over R is defined to be collection of R-modules $V[x]$ at every point $x \in S^1$ and

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morphisms $V[x] \to V[y]$ associated to any clockwise rotation from x to y with the property that rotation by 2π is multiplication by the uniformizer t of the ring R. We denote the projective representations generated at points x by $P_{[x]}$.

The Frobenius category $\mathcal{F}_\pi$ is defined in Section 2: the objects are (V, d) where V is a finitely generated projective representation of S^1 over R and d is an endomorphism of V with square equal to multiplication by t. We show that $\mathcal{F}_\pi$ is a Frobenius category which has, up to isomorphism, one indecomposable object $E(x, y)$ for every pair of points $0 \le x \le y < 2\pi$ in S^1. This is the object $P_{[x]} \coprod P_{[y]}$ with d suitably defined. The projective-injective objects are those with $x = y$, i.e $E(x, x)$. The stable category of $\mathcal{F}_\pi$ is shown to be equivalent to the continuous category $\mathcal{C}_\pi$, which is defined in 3.1.5. This construction also works in much greater generality (Proposition 3.4.1).

We also consider $\mathcal{F}_c$ for any positive real number $c \le \pi$ in 2.4; $\mathcal{F}_c$ is defined to be the additive full subcategory of $\mathcal{F}_\pi$ generated by all (x, y) where the distance from x to y is at least $\pi - c$. Objects in $\mathcal{F}_c$ are projective-injective iff they attain this minimum distance. The stable category is again triangulated and isomorphic to the continuous category $\mathcal{C}_c$ which has a cluster structure if and only if $c = (n + 1)\pi/(n + 3)$ for some positive integer n [8]. In that case we show (in [8]) that $\mathcal{C}_c$ contains a thick subcategory equivalent to the cluster category of type A_n.

The most general version of Frobenius categories that we consider in this paper, are the categories $\mathcal{F}_\phi$, for homeomorphisms $\phi : S^1 \to S^1$ satisfying "orientation preserving" and some other conditions (see 2.4.1). The categories $\mathcal{F}_c$, and in particular $\mathcal{F}_\pi$, are special cases of $\mathcal{F}_\phi$.

In later papers we will develop other properties of these continuous cluster categories. We will give recognition principle for (the morphisms in) distinguished triangles in the continuous cluster categories and other continuous categories. An example of this is given in 3.3.1. We will also show in later papers that the continuous cluster category $\mathcal{C}_\pi$ has a unique cluster up to isomorphism and we will find conditions to make the cluster character into a continuous function.

The first author would like to thank Maurice Auslander for explaining to him that "The Krull-Schmidt Theorem is a statement about endomorphism rings of objects" in a category. This observation will be used many times. Also, Maurice told us that each paper should have only one main result. So, other results will be in other papers.

0. Some remarks on topological R-categories

We recall the definition of a topological category since our constructions are motivated by our desire to construct continuously triangulated topological categories of type A. By a "continuously triangulated" category we mean a topological category which is also triangulated so that the defining equivalence T of the triangulated category is a continuous functor. We also review an easy method for defining the topology on an additive category out of the topology of a full subcategory of indecomposable objects.

Recall that a topological ring is a ring R together with a topology on R so that its structure maps are continuous. Thus addition $+ : R \times R \to R$ and multiplication $\cdot : R \times R \to R$ are required to be continuous mappings. We may sometimes also require the inverse mapping $u \mapsto u^{-1}$ to be continuous on the group of units of R. A *topological R-module* is an R-module M together with a topology on M so that the structure maps $m : R \times M \to M$ and $a : M \times M \to M$ given by $m(r, x) = rx$ and $a(x, y) = x + y$ are continuous mappings.

Definition 0.0.1. If R is a topological ring, a *topological R-category* is defined to be a small R-category $\mathcal{C}$ together with a topology on the set of objects $Ob(\mathcal{C})$ and on the set of all morphisms $Mor(\mathcal{C})$ so that the structure maps of $\mathcal{C}$ are continuous mappings. Thus s, t, id, a, m, c are continuous where

(1) $s, t : Mor(\mathcal{C}) \to Ob(\mathcal{C})$ are the source and target maps.
(2) $m : R \times Mor(\mathcal{C}) \to Mor(\mathcal{C})$, $a : A \to Mor(\mathcal{C})$ are the mappings which give the R-module structure on each hom set $\mathcal{C}(X, Y) = (s, t)^{-1}(X, Y)$. Here A is the subset of $Mor(\mathcal{C})^2$ consisting of pairs (f, g) of morphisms with the same source and target.
(3) $id : Ob(\mathcal{C}) \to Mor(\mathcal{C})$ is the mapping which sends each $X \in Ob(\mathcal{C})$ to $id_X \in \mathcal{C}(X, X) \subseteq Mor(\mathcal{C})$.
(4) $c : Mor(\mathcal{C}) \oplus Mor(\mathcal{C}) \to Mor(\mathcal{C})$ is composition and $Mor(\mathcal{C}) \oplus Mor(\mathcal{C})$ is the subset of $Mor(\mathcal{C}) \times Mor(\mathcal{C})$ on which composition is defined.

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ between topological categories $\mathcal{C}, \mathcal{D}$ is *continuous* if it is continuous on objects and morphisms. Thus, we require $Ob(F) : Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and $Mor(F) : Mor(\mathcal{C}) \to Mor(\mathcal{D})$ to be continuous mappings. When $\mathcal{C}, \mathcal{D}$ are topological R-categories, we usually assume that F is *R-linear* in the sense that the induced mappings $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ are homomorphisms of R-modules for all $X, Y \in Ob(\mathcal{C})$.

In this paper we will construct Krull-Schmidt categories $\mathcal{C}$ which have a natural topology on a full subcategory $\mathcal{D} = Ind\,\mathcal{C}$ of chosen representatives of the indecomposable objects. By the following construction, this will define a topology on the category $add\,\mathcal{D}$ consisting of direct sums of these chosen objects. This will be a small topological category equivalent to the entire category $\mathcal{C}$.

Definition 0.0.2. A topological R-category $\mathcal{D}$ is called *additive* if there is a continuous functor $\oplus : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ which is algebraically a direct sum operation. ($\mathcal{D} \times \mathcal{D}$ is given the product topology on object and morphism sets.)

Suppose $\mathcal{D}$ is a topological R-category which does not contain a zero object (and is thus not additive). Then we define the *additive category $add\,\mathcal{D}$* generated by $\mathcal{D}$ to be the category of formal ordered direct sums of objects in $\mathcal{D}$. Thus the object space of $add\,\mathcal{D}$ is:

$$Ob(add\,\mathcal{D}) = \coprod_{n \geq 0} Ob(\mathcal{D})^n$$

When $n = 0$, $Ob(\mathcal{D})^0$ consists of a single object which is the unique zero object of $add\,\mathcal{D}$. This is a topological space since it is the disjoint union of Cartesian products of topological spaces. We write the object (X_i) as the ordered sum $\coprod_i X_i$.

The morphism space is defined analogously:

$$Mor(add\,\mathcal{D}) = \coprod_{n,m \geq 0} \{((Y_i), (f_{ij}), (X_j)) \in Ob(\mathcal{D})^n \times Mor(\mathcal{D})^{nm} \times Ob(\mathcal{D})^m \mid f_{ij} \in \mathcal{D}(X_j, Y_i)\}$$

This has the topology of a disjoint union of subspaces of Cartesian products of topological spaces.

Proposition 0.0.3. *$add\,\mathcal{D}$ is a topological additive R-category in which direct sum $\oplus$ is strictly associative and has a strict unit.* □

1. Representations of the circle S^1

In this section we describe the category of representations of the circle over a discrete valuation ring. Special kinds of finitely generated projective representations of the circle will be used in section 2 in order to define Frobenius categories. Let R be a discrete valuation ring with uniformizing parameter t (a fixed generator of the unique maximal ideal $\mathfrak{m}$), quotient field $K = R/\mathfrak{m} = R/(t)$.

1.1. Representations of S^1.

Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $x \in \mathbb{R}$ and let $[x]$ denote the corresponding element $[x] = x + 2\pi\mathbb{Z}$ in S^1. When we take an element $[x] \in S^1$ we mean choose an element of S^1 and choose an arbitrary representative x of this element in $\mathbb{R}$.

Definition 1.1.1. A *representation* V of S^1 over R is defined to be:

(a) an R-module $V[x]$ for every $[x] \in S^1$ and
(b) an R-linear map $V^{(x,\alpha)} : V[x] \to V[x-\alpha]$ for all $[x] \in S^1$ and $\alpha \in \mathbb{R}_{\geq 0}$ satisfying the following conditions for all $[x] \in S^1$:
  (1) $V^{(x-\beta,\alpha)} \circ V^{(x,\beta)} = V^{(x,\alpha+\beta)}$ for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$,
  (2) $V^{(x,2\pi n)} : V[x] \to V[x]$ is multiplication by t^n for all $n \in \mathbb{Z}_{\geq 0}$.

Definition 1.1.2. A *morphism* $f : V \to W$ consists of R-linear maps $f_{[x]} : V[x] \to W[x]$ for all $[x] \in S^1$ so that $W^{(x,\alpha)} f_{[x]} = f_{[x-\alpha]} V^{(x,\alpha)}$ for all $[x] \in S^1$ and $\alpha \geq 0$.

$$\begin{array}{ccc} V[x] & \xrightarrow{f_{[x]}} & W[x] \\ \downarrow{\scriptstyle V^{(x,\alpha)}} & & \downarrow{\scriptstyle W^{(x,\alpha)}} \\ V[x-\alpha] & \xrightarrow{f_{[x-\alpha]}} & W[x-\alpha] \end{array}$$

A morphism $f : V \to W$ is called a *monomorphism* or *epimorphism* if $f_{[x]} : V[x] \to W[x]$ are monomorphisms or epimorphisms, respectively, for all $[x] \in S^1$.

Definition 1.1.3. Let $P_{[x]}$, for $[x] \in S^1$, be the representation of S^1 defined as:

(a) R-module $P_{[x]}[x-\alpha] := Re_x^\alpha$, the free R-module on one generator e_x^α for each real number $0 \leq \alpha < 2\pi$.
(b) R-homomorphism $P_{[x]}^{(x-\alpha,\beta)} : P_{[x]}[x-\alpha] \to P_{[x]}[x-\alpha-\beta]$ is the unique R-linear homomorphism defined by $P_{[x]}^{(x-\alpha,\beta)}(e_x^\alpha) = e_x^{\alpha+\beta}$. Here $e_x^{\gamma+2\pi n} := t^n e_x^\gamma \in P_{[x]}[x-\gamma]$ for $n \in \mathbb{Z}_{\geq 0}$, and $\gamma \in \mathbb{R}_{\geq 0}$ by definition.

Remark 1.1.4. It follows from the definition that e_x^0 is a generator of the representation $P_{[x]}$; we will often denote this generator by e_x.

Proposition 1.1.5. *Let V be a R-representation of S^1. There is a natural isomorphism*

$$\mathcal{P}_{S^1}(P_{[x]}, V) \cong V[x]$$

given by sending $f : P_{[x]} \to V$ to $f_{[x]}(e_x) \in V[x]$. In particular the natural ring homomorphism $R \to \mathrm{End}(P_{[x]}) \cong P_{[x]}[x] \cong R$ is an isomorphism.

Proof. Define a homomorphism $\varphi : V[x] \to \mathcal{P}_{S^1}(P_{[x]}, V)$ in the following way. For every $v \in V[x]$ let $\varphi(v) \in \mathcal{P}_{S^1}(P_{[x]}, V)$ be given by $\varphi(v)_{[x-\alpha]}(re_x^\alpha) := V^{(x,\alpha)}(rv) \in V[x-\alpha]$ for all $0 \leq \alpha < 2\pi$. Then $\varphi(v)$ is the unique morphism $P_{[x]} \to V$ such that $\varphi(v)(e_x) = v$.

In particular, $\varphi(f_{[x]}(e_x))(e_x) = f_{[x]}(e_x)) = f(e_x)$. Therefore $\varphi(f_{[x]}(e_x)) = f$ since both morphisms send the generator $e_x \in P_{[x]}[x]$ to $f_{[x]}(e_x)$. Therefore, φ gives an isomorphism $V[x] \cong \mathcal{P}_{S^1}(P_{[x]}, V)$ inverse to the map sending f to $f_{[x]}(e_x)$. □

Corollary 1.1.6. *Each representation $P_{[x]}$ is projective. In other words, if $f : V \to W$ is an epimorphism then $Hom(P_{[x]}, V) \to Hom(P_{[x]}, W)$ is surjective.* □

If $x \le y < x + 2\pi$ then $P_{[y]}[x] = R$ is generated by e_y^β where $\beta = y - x$. So, we get the following Definition/Corollary.

Definition 1.1.7. The *depth* of any nonzero morphism of the form $f : P_{[x]} \to P_{[y]}$ is defined to be the unique nonnegative real number $\delta(f) = \alpha$ so that $f(e_x) = ue_y^\alpha$ for a unit $u \in R$. We define the depth of the zero morphism to be ∞.

Lemma 1.1.8. *The depth function has the following properties.*

1. *For morphisms $f : P_{[x]} \to P_{[y]}$, $g : P_{[y]} \to P_{[z]}$ we have $\delta(g \circ f) = \delta(g) + \delta(f)$.*
2. *Given two morphism $f, g : P_{[x]} \to P_{[y]}$ and $r, s \in R$ we have $\delta(rf+sg) \ge \min(\delta(f), \delta(g))$.*

Proof. (1) If $f(e_x) = ue_y^\alpha$ and $g(e_y) = ve_z^\beta$ for units $u, v \in R$ then $gf(e_x) = uve_z^{\alpha+\beta}$ making $\delta(g \circ f) = \alpha + \beta = \delta(f) + \delta(g)$.
(2) Suppose that $f(e_x) = ue_y^\alpha$ and $g(e_x) = ve_y^\beta$ where u, v are units in R. Suppose $\alpha = \delta(f) \le \beta = \delta(g)$. Then $\beta = \alpha + 2\pi n$ for some nonnegative integer n and $g(e_x) = vt^n e_x^\alpha$. So, $rf + sg = (ru + svt^n)e_x^\alpha$ which has depth $\ge \alpha = \min(\delta(f), \delta(g))$. □

We extend the definition of depth to any morphism $f : \coprod_i P_{[x_i]} \to \coprod_j P_{[y_j]}$ by

$$\delta(f) = min\{\delta(f_{ji}) \mid f_{ji} : P_{[x_i]} \to P_{[y_j]}\}.$$

Proposition 1.1.9. *The extended notion of depth satisfies the following conditions.*

1. *Let $f : \coprod_i P_{[x_i]} \to \coprod_j P_{[y_j]}$ and $g : \coprod_j P_{[y_j]} \to \coprod_k P_{[z_k]}$. Then $\delta(g \circ f) \ge \delta(g) + \delta(f)$.*
2. *The depth of f is independent of the choice of decompositions of the domain and range of f, i.e. $\delta(f) = \delta(\psi \circ f \circ \varphi)$ for all automorphisms ψ, φ of $\coprod_j P_{[y_j]}, \coprod_i P_{[x_i]}$.*

Proof. (1) implies (2) since $\delta(\psi f \varphi) \ge \delta(\psi) + \delta(f) + \delta(\varphi) \ge \delta(f)$ and $\delta(f) \ge \delta(\psi f \varphi)$ by symmetry. Therefore, it suffices to prove (1).

By the extended definition of depth, $\delta(gf)$ is equal to the depth of one of its component functions $(gf)_{ki} : P_{[x_i]} \to P_{[z_k]}$. But this is the sum of composite functions of the form $g_{kj} f_{ji} : P_{[x_i]} \to P_{[y_j]} \to P_{[z_k]}$. By the lemma above, this gives

$$\delta(gf) = \min(\delta((gf)_{ki})) \ge \min(\delta(g_{kj}) + \delta(f_{ji})) \ge \delta(g) + \delta(f).$$ □

1.2. Finitely generated projective representations of S^1.

It is shown here that finitely generated projective representations of S^1 are precisely the finitely generated torsion free representations.

Definition 1.2.1. A representation V is *torsion-free* if each $V[x]$ is a torsion-free R-module and each map $V^{(x,\alpha)} : V[x] \to V[x - \alpha]$ is a monomorphism. A representation V is *finitely generated* if it is a quotient of a finite sum of projective modules of the form $P_{[x]}$, i.e. there exists an epimorphism $\coprod_{i=0}^n P_{[x_i]} \twoheadrightarrow V$.

We need the following lemma to prove this section and another theorem later.

Lemma 1.2.2. *Suppose V is a finitely generated torsion-free representation of S^1 and let*

$$x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = x_0 + 2\pi, \quad x_i \in \mathbb{R}.$$

For each $0 \le i \le n$ let $\{v_{ij} : j = 1, \cdots, m_i\}$ be a subset of $V[x_i]$ which maps isomorphically to a basis of the cokernel of $V^{(x_{i+1}, x_{i+1} - x_i)} : V[x_{i+1}] \to V[x_i]$ considered as a vector space over $K = R/(t)$. Let $f_{ij} : P_{[x_i]} \to V$ be the morphism defined by $f_{ij}(e_{x_i}) = v_{ij} \in V[x_i]$ Then

(1) $f = \sum_{i=0}^{n} \sum_{j=0}^{m_i} f_{ij} : P = \coprod_{i=0}^{n} m_i P_{[x_i]} \to V$ *is a monomorphism.*
(2) $f_{[x_i]} : P[x_i] \to V[x_i]$ *is an isomorphism for each* i.

Proof. Since V is torsion-free, the maps $V^{(x_i, x_i - x_0)} : V[x_i] \to V[x_0]$ are monomorphisms for $i = 0, 1, 2, \cdots, n$. Let $V_i = image(V^{(x_i, x_i - x_0)}) \subset V[x_0]$. Then

$$tV_0 = V_{n+1} \subseteq V_n \subseteq \cdots \subseteq V_2 \subseteq V_1 \subseteq V_0.$$

Furthermore, $V[x_i] \cong V_i$ and this isomorphism induces an isomorphism of quotients: $V[x_i]/V[x_{i+1}] \cong V_i/V_{i+1}$. Let $w_{ij} \in V_i \subseteq V_0$ be the image of $v_{ij} \in V[x_i]$ and let $\overline{w}_{ij} = w_{ij} + V_{i+1} \in V_i/V_{i+1} \cong V[x_i]/V[x_{i+1}]$. For each i, the $\overline{w}_{ij}$ form a basis for V_i/V_{i+1}. Taken together, $w_{ij} + tV_0$ form a basis for V_0/tV_0. Since V_0 is torsion free, it follows from Nakayama's Lemma, that the w_{ij} generate V_0 freely. Therefore, the morphism $f : P = \coprod m_i P_{x_i} \to V$ which maps the generators of P to the elements v_{ij} induces an isomorphism $f_{[x_0]} : P[x_0] \cong V[x_0]$. Applying the same argument to the points

$$x_i < x_{i+1} < \cdots < x_n < x_0 + 2\pi, x_1 + 2\pi < \cdots < x_i + 2\pi, \quad x_i \in \mathbb{R}$$

we see that $f_{[x_i]} : P[x_i] \to V[x_i]$ is an isomorphism for all i. This proves the second condition. The first condition follows. □

Proposition 1.2.3. *Every finitely generated projective representation of* S^1 *is torsion-free. Conversely, every finitely generated torsion-free representation of* S^1 *over* R *is projective and isomorphic to a direct sum of the form* $\coprod_{i=0}^{n} P_{[x_i]}$.

Proof. The first statement is clear since indecomposable projectives are torsion free and every direct summand of a torsion-free representation is torsion-free. For the second statement, let V be a finitely generated torsion-free representation of S^1. Suppose that V is generated at $n+1$ points on the circle: $[x_0], [x_1], \cdots, [x_n] \in S^1$ where $x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = x_0 + 2\pi$, with $x_i \in \mathbb{R}$, as in the lemma. Then, choose any basis $\{b_{ij}\}$ for each $V[x_i]/V[x_{i+1}]$ and lift these elements arbitrarily to elements $v_{ij} \in V[x_i]$. Let $f : P = \coprod_{i=0}^{n} m_i P_{[x_i]} \to V$ be the monomorphism given by the lemma. Then $f : P \to V$ is also onto by condition (2) in the lemma since V is generated at the points $[x_i]$. Therefore, $P \cong V$ as claimed. □

1.3. The category $\mathcal{P}_{S^1}$.

Let $\mathcal{P}_{S^1}$ be the category of all finitely generated projective (and thus torsion-free) representations of S^1 over R. By the proposition above, each indecomposable object of $\mathcal{P}_{S^1}$ is isomorphic to $P_{[x]}$ for some $[x] \in S^1$.

Lemma 1.3.1. *Any nonzero morphism* $f : P_{[x]} \to P_{[y]}$ *is a categorical epimorphism in* $\mathcal{P}_{S^1}$ *in the sense that, for any two morphisms* $g, h : P_{[y]} \to V$ *in* $\mathcal{P}_{S^1}$, $gf = hf$ *implies* $g = h$.

Proof. Let $f(e_x) = re_y^{\alpha}$ for $r \neq 0 \in R$. Then $gf(e_x) = rV^{(y,\alpha)}(g(e_y)) = hf(e_x) = rV^{(y,\alpha)}(h(e_y))$. Since V is torsion-free, this implies that $g(e_y) = h(e_y)$ making $g = h$. □

We need one more easy observation using the depth $\delta(f)$ from Definition 1.2.1.

Proposition 1.3.2. *Let* $f : P_{[x]} \to P_{[y]}$.

(1) *If* $g : P_{[x]} \to P_{[z]}$ *is a morphism so that* $\delta(f) \leq \delta(g)$ *then there is a unique morphism* $h : P_{[y]} \to P_{[z]}$ *so that* $hf = g$.
(2) *If* $g' : P_{[w]} \to P_{[y]}$ *is a morphism with* $\delta(g') \geq \delta(f)$ *then there is a unique* $h' : P_{[w]} \to P_{[x]}$ *so that* $fh' = g'$.

Proof. We prove the first statement. The second statement is similar. Let $\alpha = \delta(f), \beta = \delta(g) - \alpha$. Then $f(e_x) = re_y^\alpha$ and $g(e_x) = se_z^{\alpha+\beta}$ where r, s are units in R. Let $h : P_{[y]} \to P_{[z]}$ be the morphism given by $h(e_y) = r^{-1}se_z^\beta$. Then $hf(e_x) = rh(e_y^\alpha) = se_z^{\alpha+\beta}$. So $hf = g$. $\square$

Definition 1.3.3. Since $\mathcal{P}_{S^1}$ has one indecomposable object $P_{[x]}$ for every $[x] \in S^1$, the full subcategory $Ind\,\mathcal{P}_{S^1}$ of these objects has a natural topology. The space of objects of $Ind\,\mathcal{P}_{S^1}$ is homeomorphic to the circle S^1 and the space of morphisms is the quotient space:

$$Mor(Ind\,\mathcal{P}_{S^1}) = \{(r, x, y) \in R \times \mathbb{R} \times \mathbb{R} \mid x \leq y \leq x + 2\pi\}/\sim$$

where the equivalence relation is given by $(r, x, y) \sim (r, x + 2\pi n, y + 2\pi n)$ for any integer n and $(r, x, x + 2\pi) \sim (tr, x, x)$. Here (r, x, y) represents the morphism $P_{[x]} \to P_{[y]}$ which sends e_x to re_y^{y-x}. The second relation comes from the identity $re_x^{2\pi} = tre_x^0$. We give R the $\mathfrak{m}$-adic topology.

The category $\mathcal{P}_{S^1}$ is algebraically equivalent to the topological additive R-category $add\,Ind\,\mathcal{P}_{S^1}$ given by Definition 0.0.2.

2. The Frobenius categories $\mathcal{F}_\pi$, $\mathcal{F}_c$, $\mathcal{F}_\phi$

2.1. **Frobenius category $\mathcal{F}_\pi$.**

We define the category $\mathcal{F}_\pi$ and the set of exact sequences in $\mathcal{F}_\pi$ (and hence proper monomorphisms which are the beginning maps and proper epimorphisms which are the end maps in these exact sequences). Then we show that $\mathcal{F}_\pi$ is an exact category. Then we show that it has enough projectives with respect to the exact structure. Finally, we show that projective and injective objects in $\mathcal{F}_\pi$ coincide proving that $\mathcal{F}_\pi$ is a Frobenius category.

Definition 2.1.1. The category $\mathcal{F}_\pi$ and the exact sequences in $\mathcal{F}_\pi$ are defined as:

1. *Objects* of $\mathcal{F}_\pi$ are pairs (V, d) where $V \in \mathcal{P}_{S^1}$ and $d : V \to V$ is an endomorphism of V so that $d^2 = t$ (multiplication by t).
2. *Morphisms* in $\mathcal{F}_\pi$ are $f : (V, d) \to (W, d)$ where $f : V \to W$ satisfies $fd = df$.
3. *Exact sequences* in $\mathcal{F}_\pi$ are $(X, d) \xrightarrow{f} (Y, d) \xrightarrow{g} (Z, d)$ where $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact (and therefore split exact) in $\mathcal{P}_{S^1}$.

Remark 2.1.2. Note that if (V, d) is an object in $\mathcal{F}_\pi$, then V cannot be indecomposable since $\text{End}(P_x) = R$ does not contain an element whose square is t. We will see later that V must have an even number of components.

Theorem 2.1.3. *The category $\mathcal{F}_\pi$ is a Frobenius category.*

Lemma 2.1.4. *$f : (V, d) \to (W, d)$ is a proper monomorphism (the beginning of an exact sequence) if and only if $f : V \to W$ is a split monomorphism in $\mathcal{P}_{S^1}$. Similarly, f is a proper epimorphism in $\mathcal{F}_\pi$ if and only if it is an epimorphism in $\mathcal{P}_{S^1}$. In particular, all epimorphisms in $\mathcal{F}_\pi$ are proper.*

Proof. By definition of exactness, the split monomorphism condition is necessary. Conversely, suppose that $f : V \to W$ is split mono in $\mathcal{P}_{S^1}$. Then the cokernel C is projective, being a summand of the projective object W. Since $fd = df$, we have an induced map $d : C \to C$. Since $d^2 = t$ on V and W we must have $d^2 = t$ on C. Therefore, f is the beginning of the exact sequence $(V, d) \to (W, d) \to (C, d)$. The other case is similar with the added comment that all epimorphisms in $\mathcal{P}_{S^1}$ are split epimorphisms. $\square$

Lemma 2.1.5. *$\mathcal{F}_\pi$ is an exact category.*

Proof. We verify the dual of the short list of axioms given by Keller [10]. The first two axioms follow immediately from the lemma above.

(E0) $0 \to 0$ is a proper monomorphism.

(E1) The collection of proper monomorphisms is closed under composition.

(E2) The pushout of an exact sequence $(A, d) \xrightarrow{f} (B, d) \xrightarrow{g} (C, d)$ along any morphism $h : (A, d) \to (A', d)$ exists and gives an exact sequence $(A', d) \to (B', d) \to (C, d)$.

Pf: Since $f : A \to B$ is a split monomorphism in $\mathcal{P}_{S^1}$, so is $(f, h) : A \to B \coprod A'$. By the Lemma, we can let $(B', d) \in \mathcal{F}_\pi$ be the cokernel of (f, h). Since the pushout of a split sequence is split, the sequence $A' \to B' \to C$ splits in $\mathcal{P}_{S^1}$. Therefore $(A', d) \to (B', d) \to (C, d)$ is an exact sequence in $\mathcal{F}_\pi$. Similarly, we have the dual axiom:

(E2)^{op} The pullback of an exact sequence in $\mathcal{F}_\pi$ exists and is exact.

Therefore, $\mathcal{F}_\pi$ is an exact category. □

We record the following easy extension of this lemma for future reference.

Proposition 2.1.6. *Suppose that $\mathcal{A}$ is an additive full subcategory of $\mathcal{F}_\pi$ with the property that any proper monomorphism in $\mathcal{F}_\pi$ with both objects in $\mathcal{A}$ has cokernel in $\mathcal{A}$ and that any proper epimorphism in $\mathcal{F}_\pi$ with both objects in $\mathcal{A}$ has kernel in $\mathcal{A}$. Then $\mathcal{A}$ is an exact subcategory of $\mathcal{F}_\pi$.*

Proof. Under the first condition, proper monomorphisms in $\mathcal{A}$ will be closed under composition and under pushouts since the middle term of the pushout of $X \to Y \to Z$ under any morphism $X \to X'$ in $\mathcal{A}$ is the cokernel of the proper monomorphism $X \to Y \coprod X'$. Dually, proper epimorphisms will be closed under pull-backs since any kernel is the pull-back of a proper epimorphism. So, $\mathcal{A}$ will be exact. □

Definition 2.1.7. Let P be an object of $\mathcal{P}_{S^1}$. We define the object $P^2 \in \mathcal{F}_\pi$ to be

$$P^2 := \left(P \coprod P, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right).$$

It is clear that $(P \coprod Q)^2 = P^2 \coprod Q^2$.

The functor $P \mapsto P^2$ is both left and right adjoint to the forgetful functor $(V, d) \mapsto V$.

Lemma 2.1.8. *$\mathcal{F}_\pi(P^2, (V, d)) \cong \mathcal{P}_{S^1}(P, V)$ and, therefore, P^2 is projective in $\mathcal{F}_\pi$.*

Proof. A morphism $P^2 \to (V, d)$ is the same as a pair of morphisms $f, g : P \to V$ so that $g = df$. So, $(f, df) \leftrightarrow f$ gives the desired isomorphism. Given any epimorphism $(V, d) \to (W, d)$ and a morphism $(f, df) : P^2 \to (W, d)$, we can choose a lifting $\tilde{f} : P \to V$ of $f : P \to W$ to get a lifting $(\tilde{f}, d\tilde{f})$ of (f, df). Therefore P^2 is projective. □

Lemma 2.1.9. *$\mathcal{F}_\pi((V, d), P^2) \cong \mathcal{P}_{S^1}(V, P)$ and, therefore, P^2 is injective for proper monomorphisms in $\mathcal{F}_\pi$.*

Proof. A morphism $(V, d) \to P^2$ is the same as a pair of morphisms $f, g : V \to P$ so that $f = gd$. Therefore, $(gd, g) \leftrightarrow g$ gives the isomorphism. If $(V, d) \to (W, d)$ is a proper monomorphism and $(gd, g) : (V, d) \to P^2$ is any morphism, an extension of (gd, g) to (W, d) is given by $(\overline{g}d, \overline{g})$ where $\overline{g} : W \to P$ is an extension of $g : V \to P$ given by the assumption that $V \to W$ is a split monomorphism. Therefore, P^2 is injective for proper monomorphisms as required. □

Lemma 2.1.10. *The category $\mathcal{F}_\pi$ has enough projective and injective objects: V^2, $V \in \mathcal{P}_{S^1}$.*

Proof. For any object $(V,d) \in \mathcal{F}_\pi$ the projective-injective object V^2 maps onto (V,d) by the proper epimorphism $(1,d) : V^2 \to (V,d)$. Also $(d,1) : (V,d) \to V^2$ is a proper monomorphism. $\square$

Proof that $\mathcal{F}_\pi$ is Frobenius. There is only one thing left to prove. We need to show that every relatively projective object in $\mathcal{F}_\pi$ is isomorphic to an object of the form P^2 for some $P \in \mathcal{P}_{S^1}$ and is therefore relatively injective. Note that all relatively projective objects in $\mathcal{F}_\pi$ are projective since all epimorphisms in $\mathcal{F}_\pi$ are proper.

Let (V,d) be a projective object in $\mathcal{F}_\pi$. Then the epimorphism $(1,d) : V^2 \to (V,d)$ splits. Therefore, (V,d) is isomorphic to a direct summand of V^2. By Proposition 1.2.3, the representation V decomposes as $V \cong \coprod_{i=0}^n P_{[x_i]}$. It follows that $V^2 \cong \coprod_{i=0}^n P^2_{[x_i]}$. Therefore, (V,d) is a direct summand of $\coprod_{i=0}^n P^2_{[x_i]}$. We need a Krull-Schmidt theorem to let us conclude that (V,d) is isomorphic to a direct sum of a subset of the projective objects $P^2_{[x_i]}$. This follows from the following lemma. $\square$

Lemma 2.1.11. *The endomorphism ring of $P^2_{[x]}$ is a commutative local ring. Therefore, every indecomposable component of $\coprod P^2_{[x_i]}$ is isomorphic to one of the terms $P^2_{[x_i]}$.*

Proof. By the two previous lemmas, an endomorphism of $P^2_{[x]}$ is given by morphism

$$\begin{bmatrix} a & tb \\ b & a \end{bmatrix} : P \coprod P \to P \coprod P$$

where $a, b \in \mathrm{End}(P_{[x]}) = R$. Calculation shows that matrices of this form commute with each other. Those matrices with $a \in (t)$ form an ideal and, if $a \notin (t)$ then

$$\begin{bmatrix} a & tb \\ b & a \end{bmatrix}^{-1} = \begin{bmatrix} au & -tbu \\ -bu & au \end{bmatrix}$$

where u is the inverse of $a^2 - tb^2$ in R. Therefore, $\mathrm{End}_{\mathcal{F}_\pi}(P^2_{[x]})$ is local. $\square$

2.2. Indecomposable objects in $\mathcal{F}_\pi$.

We now describe representations $E(x,y)$ and prove that all indecomposable objects of $\mathcal{F}_\pi$ are isomorphic to these representations.

Definition 2.2.1. Let $[x], [y]$ be two (not necessarily distinct) elements of S^1 and represent them by real numbers $x \le y \le x + 2\pi$. Let $\alpha = y - x, \beta = x + 2\pi - y$ and let

$$E(x,y) = \left(P_{[x]} \coprod P_{[y]},\ d = \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right)$$

where $\alpha_* : P_{[x]} \to P_{[y]}$ is the morphism such that $\alpha_*(e_x) = e_y^\alpha$ for the generator $e_x \in P_{[x]}[x]$ and $e_y^\alpha \in P_{[y]}[x]$ and, similarly, $\beta_* : P_{[y]} \to P_{[x]}$ sends $e_y \in P_{[y]}[y]$ to $e_x^\beta \in P_{[x]}[y]$. In other words, $d(re_x^\gamma, se_y^\delta) = (se_x^{\delta+\beta}, re_y^{\gamma+\alpha})$ for all $r, s \in R$ and $\gamma, \delta \ge 0$.

There is an isomorphism $E(x,y) \cong E(y, x+2\pi)$ given by switching the two summands and an equality $E(x,y) = E(x + 2\pi n, y + 2\pi n)$ for every integer n. In the special case $x = y$, we have $\alpha = 0$ making α_* the identity map on $P_{[x]}$ and $\beta = 2\pi$ making β_* equal to multiplication by t. Thus, $E(x,x) = P^2_{[x]}$ and $E(x, x+2\pi) \cong P^2_{[x]}$ which is projective in $\mathcal{F}_\pi$.

Lemma 2.2.2. *The endomorphism ring of $E(x,y)$ is a commutative local ring. Therefore, $E(x,y)$ is an indecomposable object of $\mathcal{F}_\pi$.*

Proof. Computation shows that endomorphisms of $E(x,y)$ are given by matrices $\begin{bmatrix} a & tb \\ b & a \end{bmatrix}$ with $a, b \in R$. Therefore $\mathrm{End}_{\mathcal{F}_\pi}(E(x,y))$ is a commutative local ring as in Lemma 2.1.11. $\square$

In order to prove that the category $\mathcal{F}_\pi$ is Krull-Schmidt we need the following lemma, which uses the notion of depth as defined in 1.1.7

Lemma 2.2.3. *Let (V,d) be an object in $\mathcal{F}_\pi$ and let $\varphi : V \cong \coprod_{i=0}^n P_{[x_i]}$ be a decomposition of V into indecomposable summands. Let $f_{ji} : P_{[x_i]} \to P_{[x_j]}$ be the component of $f = \varphi d \varphi^{-1}$ with the smallest depth. Then we may choose $i \neq j$, and the representatives $x_i, x_j \in \mathbb{R}$ so that $x_i \le x_j \le x_i + \pi$ and $E(x_i, x_j)$ is a direct summand of (V,d).*

Proof. We first note that, since the depth of $d^2 = t$ is 2π, the depth of d is $\delta(d) \le \pi$. Therefore $\delta(f) \le \pi$. Next, we show that the minimal depth is attained by an off-diagonal entry of the matrix $(f_{ji} : P_{[x_i]} \to P_{[x_j]})$. Suppose that a diagonal entry f_{ii} has the minimal depth. Then $\delta(f_{ii}) = 0$ (since it can't be 2π). But then f_{ii} is an isomorphism. But f^2 is zero modulo t. To cancel the f_{ii}^2 term in f^2 there must be some $j \neq i$ so that f_{ji} is also an isomorphism, making $\delta(f_{ji}) = 0$.

So, we may assume that $P_{[x_i]}$ and $P_{[x_j]}$ are distinct components of V and we may choose the representatives x_i, x_j in $\mathbb{R}$ so that $x_i \le x_j \le x_i + \pi$ and $\delta(d) = \delta(f) = \delta(f_{ji}) = x_j - x_i$. Let $\alpha = x_j - x_i$ and $\beta = x_i + 2\pi - x_j = 2\pi - \alpha$. We now construct a map $\rho : E(x_i, x_j) \to V$,

$$\left(P_{[x_i]} \coprod P_{[x_j]}, d_E = \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right) \xrightarrow{\rho} (V,d);$$

$\alpha_* : P_{[x_i]} \to P_{[x_j]}$ is defined by $\alpha_*(e_{x_i}) = e^\alpha_{x_j}$ and $\beta_* : P_{[x_j]} \to P_{[x_i]}$ by $\beta_*(e_{x_j}) = e^\alpha_{x_i}$. So $\delta(\alpha_*) = \alpha$ and $\delta(\beta_*) = \beta$. In order to define ρ consider the following diagram where top squares commute by the definition of f. The existence and uniqueness of the map $h : P_{[x_j]} \to \coprod_k P_{[x_k]}$ follows by Proposition 1.3.2 since $\delta(\alpha_*) = \alpha \le \delta(f \circ incl_i)$.

$$\begin{array}{ccccc}
V & \xrightarrow{d} & V & \xrightarrow{d} & V \\
\varphi \downarrow \cong & & \varphi \downarrow \cong & & \varphi \downarrow \cong \\
\coprod_k P_{[x_k]} & \xrightarrow{f} & \coprod_k P_{[x_k]} & \xrightarrow{f} & \coprod_k P_{[x_k]} \\
incl_i \uparrow & & \exists! h \uparrow & & incl_i \uparrow \\
P_{[x_i]} & \xrightarrow{\alpha_*} & P_{[x_j]} & \xrightarrow{\beta_*} & P_{[x_i]}
\end{array}$$

Notice that (1) $f^2 incl_i = incl_i \beta_* \alpha_*$ since both maps are multiplications by t; reasons: $f^2 = \varphi d^2 \varphi^{-1}$ and therefore is multiplication by t, and since $\delta(\beta_* \alpha_*) = 2\pi$ the map $\beta_* \alpha_*$ is also multiplication by t. From $f\, incl_i = h \alpha_*$ and (1) it follows that $f h \alpha_* = incl_i \beta_* \alpha_*$. Since α_* is a categorical epimorphism in $\mathcal{P}_{S^1}$, it follows that the bottom right square commutes, i.e. $fh = incl_i \beta_* = fh$.

Define $\rho := (\varphi^{-1} incl_i, \varphi^{-1} h)$ and check that $d \circ \rho = \rho \circ d_E$. Then $d \circ \rho = (d\varphi^{-1} incl_i, d\varphi^{-1} h)$, and $\rho \circ d_E = (\varphi^{-1} incl_i, \varphi^{-1} h) \circ d_E = (\varphi^{-1} h \circ \alpha_*, \varphi^{-1} incl_i \circ \beta_*) = (d\varphi^{-1} incl_i, d\varphi^{-1} h) = d \circ \rho$.

Similarly we get the diagram

$$\begin{array}{ccccc}
V & \xrightarrow{d} & V & \xrightarrow{d} & V \\
\varphi \downarrow \cong & & \varphi \downarrow \cong & & \varphi \downarrow \cong \\
\coprod_k P_{[x_k]} & \xrightarrow{f} & \coprod_k P_{[x_k]} & \xrightarrow{f} & \coprod_k P_{[x_k]} \\
proj_j \downarrow & & \exists ! g \downarrow & & proj_j \downarrow \\
P_{[x_j]} & \xrightarrow{\beta_*} & P_{[x_i]} & \xrightarrow{\alpha_*} & P_{[x_j]}
\end{array}$$

and the map $\rho' : (V, d) \to (P_{[x_i]} \coprod P_{[x_j]}, d_E)$ defined as $\rho' = (g\varphi, proj_j \varphi)$.Then $\rho' \circ d = d_E \circ \rho'$. Then the composition $E(x_i, x_j) \xrightarrow{\rho} (V, d) \xrightarrow{\rho'} E(x_i, x_j)$ is

$$\rho'\rho = \begin{bmatrix} g \circ incl_i & g \circ h \\ proj_j \circ incl_i & proj_j \circ h \end{bmatrix}$$

which is an isomorphism since both i-th component of g and j-th components of h are isomorphisms making the diagonal entries of this matrix invertible as in the proof of Lemma 2.1.11. So $E(x_i, x_j)$ is isomorphic to a summand of (V, d). □

Theorem 2.2.4. *The category $\mathcal{F}_\pi$ is a Krull-Schmidt category with indecomposable objects isomorphic to $E(x, y)$ for some $0 \le x \le y < 2\pi$.*

Proof. The Lemma 2.2.3 implies the theorem since it shows, by induction on the number of components of V, that (V, d) is a direct sum of indecomposable objects $E(x, y)$. □

Corollary 2.2.5. *Indecomposable projective-injective objects in $\mathcal{F}_\pi$ are isomorphic to $E(x, x)$.*

The set of isomorphism classes of indecomposable objects of $\mathcal{F}_\pi$ has a natural topology as a compact Moebius band. However, we need to keep $E(x, y)$ and $E(y, x + 2\pi)$ as separate objects in order to have an oriented manifold for reasons which we explain in the next section. This leads to the following definition.

Definition 2.2.6. Let $\widetilde{Ind}\,\mathcal{F}_\pi$ be the topological R-category whose object set is the set of all $E(x, y)$ with $x \le y \le x + 2\pi$. We give this set the topology as a quotient space of a subspace of the plane:

$$Ob(\widetilde{Ind}\,\mathcal{F}_\pi) = \{(x, y) \in \mathbb{R}^n \mid x \le y \le x + 2\pi\}/\sim$$

where the equivalence relation is $(x, y) \sim (x + 2\pi n, y + 2\pi n)$ for all $n \in \mathbb{Z}$. This is a compact Hausdorff space homeomorphic to the Cartesian product $S^1 \times [0, 2\pi]$.

The space of morphisms $Ob(\widetilde{Ind}\,\mathcal{F}_\pi)$ is not important but we specify it for completeness. It is the quotient space of the space of all 6-tuples $(r, s, (x_1, x_2), (y_1, y_2)) \in R^2 \times \mathbb{R}^4$ satisfying the following closed conditions: (1) $x_1 \le x_2 \le x_1 + 2\pi$, (2) $y_1 \le y_2 \le y_1 + 2\pi$, (3) Either $r = 0$ or $y_1 \ge x_1$ and $y_2 \ge x_2$, (4) Either $s = 0$ or $y_2 \ge x_1$ and $y_1 + 2\pi \ge x_2$. The equivalence relation is given by $(r, s, (x_1, x_2), (y_1, y_2)) \sim (r, s, (x_1 + 2\pi n, x_2 + 2\pi n), (y_1 + 2\pi n, y_2 + 2\pi n))$ for all $n \in \mathbb{Z}$ and $(r, s, (x_1, x_2), (y_1 + 2\pi, y_2 + 2\pi)) \sim (tr, ts, (x_1, x_2), (y_1, y_2))$. The morphism (r, s, X, Y) represents r times the basic diagonal morphism $X \to Y$ plus s times the basic counter-diagonal morphism. (Morphisms $X \to Y$ are given by 2×2 matrices.)

For the topological model for $\mathcal{F}_\pi$ we take $add\,\widetilde{Ind}\,\mathcal{F}_\pi$.

2.3. **Support intervals.**
We will formulate an extension of Lemma 2.2.3 which will be useful for constructing other Frobenius categories. To do this we replace depth conditions with conditions on the "support intervals" of a morphism.

A *closed interval* in S^1 is defined to be a closed subset of the form $[x, y]$ where $x \le y < x + 2\pi$. These subsets are characterized by the property that they are nonempty, compact and simply connected. For example, a single point is a closed interval.

Let $P = \coprod P_{[x_i]}$, $Q = \coprod P_{[y_j]}$ be objects of $\mathcal{P}_{S^1}$ with given decompositions into indecomposable objects. Let $f : P \to Q$ be a morphism with components $f_{ji} : P_{[x_i]} \to P_{[y_j]}$. Then $\delta(f_{ji}) = y_j - x_i + 2\pi n$ for some $n \in \mathbb{Z}_{\ge} 0$ (or $\delta(f_{ji}) = \infty$). Consider the collection of all closed intervals $[x_i, y_j] \subsetneq S^1$ with the property that $\delta(f_{ji}) = y_j - x_i$. A minimal element of this collection (ordered by inclusion) will be called a *support interval* for f. The collection of all support intervals is the *support* of f.

As an example, if $E(x, y) = (P, d)$ where $x \ne y$ then the support of d consists of the intervals $[x, y]$ and $[y, x + 2\pi]$.

Proposition 2.3.1. *Let $f : P \to Q$ be a morphism. The support of f is independent of the choice of decompositions of P and Q.*

Proof. Let $f' = \psi \circ f \circ \varphi$ where φ, ψ are automorphisms of P, Q respectively. Let $[x_i, y_j]$ be a support interval for f'. Then we have a nonzero composition:

$$P_{[x_i]} \to \coprod P_{[x_a]} \xrightarrow{f} \coprod P_{[y_b]} \to P_{[y_j]}$$

of depth $< 2\pi$. This implies that f has a support interval $[x_a, y_b]$ for some a, b and $[x_a, y_b] \subset [x_i, y_j]$. Therefore, every support interval of f' contains a support interval of f. The reverse is also true by symmetry. So, the supports of f, f' must be equal. □

Remark 2.3.2. Note that the depth $\delta(f)$ of a morphism $f : P \to Q$ in $\mathcal{P}_{S^1}$ is equal to the minimum length $y - x$ for all support intervals $[x, y]$ of f when f has nonempty support and $\delta(f) \ge 2\pi$ otherwise.

The following proposition generalizes Lemma 2.2.3 above.

Proposition 2.3.3. *Let $(V, d) \in \mathcal{F}_\pi$. Suppose that $x \le y < x + 2\pi$ and the closed interval $[x, y] \subset S^1$ does not properly contain any support interval of $d : V \to V$. Then*

1. $\mathcal{F}_\pi(E(x, y), (V, d)) \cong \mathcal{P}_{S^1}(P_{[x]}, V)$ *where the isomorphism is given by restriction to the component $P_{[x]}$ of $E(x, y)$.*
2. $\mathcal{F}_\pi((V, d), E(x, y)) \cong \mathcal{P}_{S^1}(V, P_{[y]})$ *where the isomorphism is given by projection to $P_{[y]}$.*

Proof. These statements follow from Proposition 1.3.2, as illustrated in the two diagrams in the proof of Lemma 2.2.3 above. □

2.4. **The Frobenius categories $\mathcal{F}_c, \mathcal{F}_\phi$.**
Let $c, \theta \in \mathbb{R}_{>0}$ be such that $c + \theta = \pi$ and let $\mathcal{F}_c$ denote the full subcategory of $\mathcal{F}_\pi$ whose objects are all (V, d) with the property that the depth of d is $\delta(d) \ge \theta$. We show that $\mathcal{F}_c$ is a Frobenius category whose stable category is equivalent to the category $\mathcal{C}_c$ defined in next section and discussed in detail in later papers in this series. In particular, the category $\mathcal{C}_c$ will be shown to be a cluster category (without coefficients or frozen objects) if and only if $\theta = 2\pi/(n + 3)$ for some positive integer n. The category $\mathcal{F}_c$ is a special case of the

following more general construction which produces many examples of cluster and m-cluster categories as we will show in other papers.

Definition 2.4.1. The category $\mathcal{F}_\phi$ is defined as the full subcategory of $\mathcal{F}_\pi$ consisting of all (V, d) so that every support interval $[x, y]$ of d contains an interval of the form $[z, \phi(z)]$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism of the real line to itself satisfying:

(1) $\phi(x + 2\pi) = \phi(x) + 2\pi$.
(2) $x \leq \phi(x) < x + \pi$ for all $x \in \mathbb{R}$.

The first condition implies that ϕ induces an orientation preserving homeomorphism $\overline{\phi}$ of the circle S^1 to itself. The second condition says that $\overline{\phi}$ "does not move points clockwise" and also implies that $\phi^2(x) < x + 2\pi$. The condition on the support interval $[x, y]$ is equivalent to the condition $\phi(x) \leq y$. In the special case when $\phi(x) = x + \theta$ where $\theta = \pi - c$, this condition is: $y \geq x + \theta$. So, $\mathcal{F}_\phi = \mathcal{F}_c$ in this case.

Proposition 2.4.2. *The category $\mathcal{F}_\phi$ is a Krull-Schmidt category with indecomposable objects isomorphic to $E(x, y)$ with the property that $\phi(x) \leq y$ and $\phi(y) \leq x + 2\pi$.* □

To prove that $\mathcal{F}_\phi$ in general, and $\mathcal{F}_c$ in particular, is a Frobenius category, the following observation is helpful.

Lemma 2.4.3. (1) *Suppose that $x < y < z < x + 2\pi$. Then a morphism $f : P_{[x]} \to P_{[y]}$ factors through $P_{[z]}$ if and only if $f = tg$ for some $g : P_{[x]} \to P_{[y]}$.*
(2) *If $f : E(x, z) \to E(x, y)$ is a morphism whose $P_{[x]} - P_{[x]}$ component is an isomorphism then $x \leq z \leq y$.*

Proof. If $f : P_{[x]} \to P_{[y]}$ factors through $P_{[z]}$ then its depth must be at least $y - x + 2\pi$. So, it is divisible by t. Conversely, any morphism which is divisible by t factors through $P_{[w]}$ for all points $[w] \in S^1$. This proves (1) and (1) implies (2). □

Theorem 2.4.4. *The category $\mathcal{F}_\phi$ is a Frobenius category with projective-injective objects $E(x, y)$ where either $y = \phi(x)$ or $x + 2\pi = \phi(y)$.*

Proof. To show that $\mathcal{F}_\phi$ is an exact category it suffices, by Proposition 2.1.6, to show that a proper monomorphism in $\mathcal{F}_\pi$ with both objects in $\mathcal{F}_\phi$ has cokernel in $\mathcal{F}_\phi$ and similarly for kernels. So, let $(X, d) \xrightarrow{f} (Y, d) \xrightarrow{g} (Z, d)$ be an exact sequence in $\mathcal{F}_\pi$ so that $(X, d), (Y, d)$ lie in $\mathcal{F}_\phi$ and let $E(x, y)$ be a component of (Z, d). If $E(x, y)$ is a component of (Y, d), then it is in $\mathcal{F}_\phi$. Now suppose $E(x, y)$ is a component of (Z, d) but not of (Y, d). Since $Y \to Z$ is split epimorphism in $\mathcal{P}_{S^1}$, there are components $E(x, a), E(y, b)$ of (Y, d) so that:

(a) $P_{[x]} \subseteq E(x, a)$ and $g(P_{[x]}) \cong P_{[x]} \subseteq E(x, y)$ and
(b) $P_{[y]} \subseteq E(y, b)$ and $g(P_{[y]} \cong P_{[y]} \subseteq E(x, y)$.

By Lemma 2.4.3 above, (a) implies that $x < a \leq y$. Since (Y, d) lies in $\mathcal{F}_\phi$, we must have $\phi(x) \leq a \leq y$. Similarly, (b) implies $y < b \leq x + 2\pi$. Since (Y, d) lies in $\mathcal{F}_\phi$, this implies $\phi(y) \leq b \leq x + 2\pi$. By Proposition 2.4.2 this implies that $E(x, y)$ lies in $\mathcal{F}_\phi$. A similar argument shows that any kernel of a proper epimorphism in $\mathcal{F}_\phi$ lies in $\mathcal{F}_\phi$.

To show that $E(x, y)$ is projective with respect to exact sequences in $\mathcal{F}_\phi$ for $y = \phi(x)$, suppose that $p : (Y, d_Y) \to (Z, d_Z)$ is an admissible epimorphism in $\mathcal{F}_\phi$. Since $[x, y]$ does not properly contain any support interval for either d_Y or d_Z, we have by Proposition 2.3.3 that $\mathrm{Hom}(E(x, y), (Y, d_Y) = \mathrm{Hom}(P_{[x]}, Y)$ and $\mathrm{Hom}(E(x, y), (Z, d_Z)) = \mathrm{Hom}(P_{[x]}, Z)$. Since $Y \to Z$ is split epi, any morphism $E(x, y) \to (Z, d_Z)$ lifts to (Y, d_Y). The dual argument using the second part of Proposition 2.3.3 proves that $E(x, y)$ is injective.

For any other indecomposable object $E(x,y)$ of $\mathcal{F}_\phi$ we have, by Proposition 2.4.2, that $\phi(x) \le y$ and $\phi(y) \le x + 2\pi$. So, we have admissible epimorphism and monomorphism:

$$E(x,\phi(x)) \coprod E(y,\phi(y)) \to E(x,y), \qquad E(x,y) \to E(\phi^{-1}(y),y) \coprod E(\phi^{-1}(x),x).$$

If $E(x,y)$ is projective or injective then either the first or second map is split and we get that $y = \phi(x)$. This show that we have enough projectives and enough injectives and that they all have the form $E(x,\phi(x)) \cong E(\phi(x), x+2\pi)$. □

Corollary 2.4.5. *The category $\mathcal{F}_c$ is a Frobenius category with projective-injective objects $E(x, x+\theta) \cong E(x+\theta, x+2\pi)$.* □

3. Continuous cluster categories

Categories $\mathcal{C}_\pi$ and $\mathcal{C}_c$ are defined here; we show that they are isomorphic to the stable categories of the Frobenius categories $\mathcal{F}_\pi$ and $\mathcal{F}_c$, which are triangulated by Happel's theorem. All the structure maps, including the triangulation maps, are continuous. In a subsequent paper we show that the category $\mathcal{C}_\pi$ has cluster structure, hence the name: continuous cluster category.

The stable category of a Frobenius category is triangulated and this triangulation is used to define the triangulated structure on the stable category. We will show that the stable category of $\mathcal{F}_\pi$ is isomorphic to the continuous cluster category $\mathcal{C}_\pi$ and the stable category of $\mathcal{F}_\phi$ contains the cluster category of type A_n as a thick subcategory if $\phi^{n+3}(x) = x + 2\pi$ for all x. The cluster structures of these categories are discussed in other papers.

3.1. The stable category $\underline{\mathcal{F}_\pi}$ and continuous cluster category $\mathcal{C}_\pi$.

We first recall some basic properties of the stable category $\underline{\mathcal{F}_\pi}$, then define the continuous cluster category $\mathcal{C}_\pi$. It will follow from the definition that $\underline{\mathcal{F}_\pi}$ and $\mathcal{C}_\pi$ are algebraically equivalent. There is a difference in their topologies which we will explain.

Recall that the stable category $\underline{\mathcal{F}_\pi}$ has the same objects as $\mathcal{F}_\pi$ and the morphism sets $\underline{\mathcal{F}_\pi}(X,Y)$ are quotients of $\mathrm{Hom}(X,Y)$ modulo those morphisms which factor through projective-injective objects, which are direct sums of objects of the form:

$$E(x,x) = \left(P_{[x]}^2, d = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right).$$

For example, multiplication by t is 0 in the stable category $\underline{\mathcal{F}_\pi}$ since $t = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ factors as:

$$E(x,y) = P_{[x]} \coprod P_{[y]} \xrightarrow{\begin{bmatrix} 0 & \beta_* \\ 1 & 0 \end{bmatrix}} P_{[x]} \coprod P_{[x]} \xrightarrow{\begin{bmatrix} 0 & t \\ \alpha_* & 0 \end{bmatrix}} P_{[x]} \coprod P_{[y]} = E(x,y)\ .$$

We also recall that $E(x,y)$ is isomorphic to $E(y, x+2\pi)$.

The following Lemmas will be used to prove the isomorphism on the Hom sets.

Lemma 3.1.1. *If $x < y \le a < b < x + 2\pi$ then $\underline{\mathcal{F}_\pi}(E(x,y), E(a,b)) = 0$.*

Proof. All morphisms from $P_{[x]}$ to $P_{[a]} \coprod P_{[b]}$ factor through $\alpha_* : P_{[x]} \to P_{[y]}$. So, any morphism $E(x,y) \to E(a,b)$ factors though $\alpha_* \oplus 1 : E(x,y) \to E(y,y)$. □

Lemma 3.1.2. *(a) If $x \le a < y \le b < x + 2\pi$ then $\underline{\mathcal{F}}_\pi(E(x,y), E(a,b)) = K$ is generated by $f \coprod g$ where $f : P_x \to P_a$ sends e_x to e_a^{a-x} and $g : P_y \to P_b$ sends e_y to e_b^{b-y} where $V = E(a,b)$.*

(b) Furthermore, any nonzero multiple of $f \coprod g$ factors through $E(c,d)$ if and only if either $x \le c \le a < y \le d \le b < x + 2\pi$ (for some choice of liftings of c, d to $\mathbb{R}$) or $x \le d \le a < y \le c+2\pi \le b < x+2\pi$ (the same condition with (c,d) replaced by $(d, c+2\pi)$).

Proof. Any morphism $E(x,y) \to E(a,b)$ is a sum of two morphisms: a diagonal and counter-diagonal morphism:

$$\begin{bmatrix} f & h \\ k & g \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} 0 & h \\ k & 0 \end{bmatrix}$$

The counter-diagonal morphism is stably trivial since it factors through $E(y,y)$. The morphisms f, g make the following diagram commute.

$$\begin{array}{ccc} P_x & \xrightarrow{f} & P_a \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \gamma_*} \\ P_y & \xrightarrow{g} & P_b \end{array}$$

So, if $f(e_x) = re_a^{a-x}$ and $g(e_y) = se_b^{b-y}$, then $\gamma_*(f(e_x)) = re_b^{b-x} = g(\alpha_*(e_x)) = se_b^{b-x}$ making $r = s$. If $r = s \in (t)$ then the morphism $f \coprod g$ is divisible by t and is thus stably trivial. If $r = s \notin (t)$ then neither f nor g is divisible by t. In this case, $f : P_x \to P_a$ can only factor through P_c where $x \le c \le a$ and $g : P_y \to P_b$ can only factor through P_d where $y \le d \le b$. Thus $f \coprod g$ factors through $E(c,d)$ via diagonal morphisms if and only if $x \le c \le a < y \le d \le b < x + 2\pi$ (for some choice of liftings of c, d to $\mathbb{R}$). $f \coprod g$ factors through $E(c,d)$ with counter-diagonal morphisms iff the other condition holds. Since the intervals $[x,a], [y,b]$ are disjoint on the circle S^1, $f \coprod g$ cannot factor through a projective-injective P_z^2. Therefore, $f \coprod g$ is not stably trivial. Since $r = s$, the stable hom set $\underline{\mathcal{F}}_\pi(E(x,y), E(a,b))$ is one dimensional and generated by the morphism $f \coprod g$ with $r = s = 1$ as claimed. □

Definition 3.1.3. The topology of the stable category $\underline{\mathcal{F}}_\pi$ is given as follows. The object space of $\underline{\mathcal{F}}_\pi$ is equal to the object space of $\mathcal{F}_\pi$ with the same topology. The morphism space $Mor(\underline{\mathcal{F}}_\pi)$ is the quotient space of $Mor(\mathcal{F}_\pi)$ modulo the standard equivalence relation reviewed above with the quotient topology.

Proposition 3.1.4. *There is an isomorphism of topological categories $\underline{\mathcal{F}}_\pi \cong add\, \underline{\widetilde{Ind}\ \mathcal{F}_\pi}$ where $\underline{\widetilde{Ind}\ \mathcal{F}_\pi}$ is the full subcategory of $\underline{\mathcal{F}}_\pi$ with object set equal to $Ob(Ind\, \mathcal{F}_\pi)$ with the subspace topology.* □

By definition the stable category $\underline{\mathcal{F}}_\pi$ has an infinite number of (isomorphic) zero objects. When we identify all of these objects to one point and take the quotient topology, we get the continuous cluster category. This quotient topology is the same as the James construction ([Hatcher], p. 244) on $\underline{\widetilde{Ind}\mathcal{F}_\pi}$ with the zero objects identified to one point. We will go over this in the paper where it is needed. (The main point is that the set of distinguished triangles with a bounded number of components in its objects is a compact Hausdorff space.)

Definition 3.1.5. For any field K the *continuous cluster category* $\mathcal{C}_\pi$ is the additive category generated by the K-category $\widetilde{Ind}\, \mathcal{C}_\pi$ defined as follows.

The *object set* of $\widetilde{Ind}\,\mathcal{C}_\pi$ will be the set of all ordered pairs of distinct point S^1. Objects are labeled by pairs of real numbers (x,y) with $x<y<x+2\pi$ with $(x,y)=(x+2\pi n, y+2\pi n)$ for all integers n.

Morphism set: For any two objects X,Y we define $\mathcal{C}_\pi(X,Y)$ to be a one-dimensional vector space if the coordinates $X=(x_0,x_1), Y=(y_0,y_1)$ can be chosen to satisfy either $x_0\le y_0<x_1\le y_1<x_0+2\pi$ or $x_0\le y_1<x_1\le y_0+2\pi<x_0+2\pi$. We denote the generator of $\mathcal{C}_\pi(X,Y)$ by b_+^{XY} in the first case and b_-^{XY} in the second case. For example, the identity map on X is b_+^{XX}. Nonzero morphisms are either rb_+^{XY} or rb_-^{XY} where $r\neq 0\in K$. The composition of morphisms $rb_\epsilon^{XY}:X\to Y$ and $sb_{\epsilon'}^{YZ}:Y\to Z$ is defined to be $rsb_{\epsilon\epsilon'}^{XZ}:X\to Z$ provided that $\mathcal{C}_\pi(X,Z)$ is nonzero with generator $b_{\epsilon\epsilon'}^{XZ}$. Otherwise the composition is zero. (We multiply signs by $\epsilon\epsilon'=+$ if $\epsilon=\epsilon'$ and $\epsilon\epsilon'=-$ if $\epsilon\neq\epsilon'$.) We call the morphisms b_ϵ^{XY} *basic morphisms*. Any composition of basic morphisms is either basic or zero.

Theorem 3.1.6. *The stable category of the continuous Frobenius category $\mathcal{F}_\pi$ is equivalent, as additive category, to the continuous cluster category $\mathcal{C}_\pi$, i.e. $\underline{\mathcal{F}}_\pi\approx\mathcal{C}_\pi$.*

Proof. Since both categories are Krull-Schmidt, it suffices to show that the full subcategories of indecomposable objects are equivalent. We will show that the full subcategory $\widetilde{\underline{Ind}}_0\mathcal{F}_\pi$ of $\underline{\mathcal{F}}_\pi$ with objects $E(x,y)$ for $0\le x<2\pi$ and $x<y<x+2\pi$ (i.e., the nonzero objects of $\widetilde{\underline{Ind}\ \mathcal{F}}_\pi$) is isomorphic to $\widetilde{Ind}\,\mathcal{C}_\pi$. The isomorphism

$$\Psi:\widetilde{\underline{Ind}}_0\mathcal{F}_\pi\xrightarrow{\cong}\widetilde{Ind}\,\mathcal{C}_\pi$$

is given on objects by $\Psi E(x,y)=(x,y)\in S^1\times S^1$ considered as an object of $\widetilde{Ind}\,\mathcal{C}_\pi$. This is a bijection on objects. By Lemma 3.1.1 and Lemma 3.1.2(a), Φ is an isomorphism on Hom sets. By Lemma 3.1.2(b), the composition of two basic morphisms $E(x,y)\to E(c,d)\to E(a,b)$ is a nonzero basic morphism if and only if $x\le c\le a<y\le d\le b<x+2\pi$ (for some choice of liftings to $\mathbb{R}$) or the analogous condition holds with (c,d) replaced by $(d,c+2\pi)$ or with (a,b) replaced by $(b,a+2\pi)$. So, Ψ respects composition making Ψ an isomorphism of categories. □

Remark 3.1.7. The functor Ψ induces an equivalence of categories $\underline{\mathcal{F}}_\pi\to\mathcal{C}_\pi$ which is surjective on object and morphism sets. We give $\mathcal{C}_\pi$ the quotient topology. Readers familiar with topology will recognize that this is the James construction making the object set of $\mathcal{C}_\pi$ homotopy equivalent to $\Omega\Sigma(S^2\vee S^1)$. (See [5], p. 224.)

Remark 3.1.8. The category $\widetilde{Ind}\,\mathcal{C}_\pi$ has exactly two objects in every isomorphism class. If we identify these two objects, using the quotient topology, we get a topological category $Ind\,\mathcal{C}_\pi$ whose object set is an open Moebius band. However, it is not possible to put a continuous triangulated structure on $add\,Ind\,\mathcal{C}_\pi$ (when K has characteristic different from 2 and has the discrete topology) since $X\cong TX$. If $TX=X$ then T must send all endomorphisms of X to themselves and therefore must send all morphisms to themselves since all nonzero morphisms $X\to Y$ between objects in $Ind\,\mathcal{C}_\pi$ are connected by a path to a nonzero morphism $X\to X$. This forces T to be the identity functor which is a contradiction.

This reasoning forces us to define $\widetilde{Ind}\,\mathcal{C}_\pi$ to be a two fold covering of the Moebius band. We chose the oriented connected covering. However, the unoriented disconnected covering also works. See Remark 3.4.2 below.

3.2. **Stable categories $\underline{\mathcal{F}}_c$ and $\underline{\mathcal{F}}_\phi$ and the continuous categories $\mathcal{C}_c$ and $\mathcal{C}_\phi$.**

The Frobenius categories $\mathcal{F}_c$ and $\mathcal{F}_\phi$ were defined and studied in the previous section. Here we define categories $\mathcal{C}_c$ and $\mathcal{C}_\phi$ and show that they are isomorphic to the stable categories of $\mathcal{F}_c$ and $\mathcal{F}_\phi$. The categories $\mathcal{C}_c$ and $\mathcal{C}_\phi$ are continuous triangulated categories, however they are not necessarily cluster categories.

In a subsequent paper [8] we show that $\mathcal{C}_c$ has cluster structure precisely when $c = (n+1)\pi/(n+3)$. In that case the clusters are finite and all are contained in a thick subcategory of $\mathcal{C}_c$ which is isomorphic to the cluster category of type A_n. The $\mathcal{C}_\phi$ construction is more versatile and has a cluster structure whenever ϕ has fixed points. For example, if ϕ has exactly one fixed point then $\mathcal{C}_\phi$ contains the cluster category of type A_∞ as a thick subcategory. $\mathcal{C}_\phi$ has an m-cluster structure if ϕ has exactly m periodic points of finite period $m \geq 3$ and certain other conditions are satisfied. These clusters and m-clusters will have an infinite number of objects and will be explored in other papers.

Since $\mathcal{C}_c$ are special cases of $\mathcal{C}_\phi$, we give the definition only in the second case.

Definition 3.2.1. Let K be a field and ϕ a homeomorphism of $\mathbb{R}$ as in Definition 2.4.1 above. The *continuous category* $\mathcal{C}_\phi$ is defined to be the additive category generated by $\widetilde{Ind}\,\mathcal{C}_\phi$ where $\widetilde{Ind}\,\mathcal{C}_\phi$ denotes the category with:
1. Objects are ordered pairs of points (x_0, x_1) in S^1 so that $x_0 \leq \phi(x_0) \leq x_1 \leq \phi^{-1}(x_0+2\pi)$.
2. Morphisms are given by

$$\mathcal{C}_\phi(X,Y) = \begin{cases} Kb_+^{XY} & \text{if } x_0 \leq y_0 < \phi^{-1}(x_1) \leq x_1 \leq y_1 < \phi^{-1}(x_0+2\pi) \\ Kb_-^{XY} & \text{if } x_0 \leq y_1 < \phi^{-1}(x_1) \leq x_1 \leq y_0 + 2\pi < \phi^{-1}(x_0+2\pi) \\ 0 & \text{if the elements of } X, Y \text{ do not lift to such real numbers} \end{cases}$$

3. Composition of morphisms is given by

$$rb_\epsilon^{YZ} \circ sb_{\epsilon'}^{XY} = \begin{cases} rsb_{\epsilon\epsilon'}^{XZ} & \text{if } \mathcal{C}_c(X,Z) = Kb_{\epsilon\epsilon'}^{XZ} \\ 0 & \text{otherwise} \end{cases}$$

For $0 < c \leq \pi$ the category $\mathcal{C}_c$ is defined to be $\mathcal{C}_\phi$ in the case when $\phi(x) = x + \pi - c$ for all $x \in \mathbb{R}$.

Proposition 3.2.2. *The stable category of the Frobenius category $\mathcal{F}_\phi$ is equivalent to the category $\mathcal{C}_\phi$.*

Proof. The verification of the proposition follows the pattern of Theorem 3.1.6 and is straightforward. □

Corollary 3.2.3. *For any positive $c < \pi$, the stable category of the Frobenius category $\mathcal{F}_c$ is equivalent to the category $\mathcal{C}_c$.*

Remark 3.2.4. The category $\mathcal{F}_\phi$ is a subcategory of $\mathcal{F}$ and we have functors $\mathcal{F}_\phi \to \underline{\mathcal{F}}_\phi \to \mathcal{C}_\phi$ which are epimorphisms on objects and morphisms. Thus, we can give $\mathcal{F}_phi$ the subspace topology with respect to $\mathcal{F}$ and $\underline{\mathcal{F}}_\phi$, $\mathcal{C}_\phi$ the quotient topologies with respect to $\underline{\mathcal{F}}_\phi$. This, of course, applies to the special case of $\mathcal{F}_c$ and $\mathcal{C}_c$.

3.3. **Distinguished triangles.** In order to obtain an explicit triangulation of the stable category of any Frobenius category we need to fix a choice, for each object X, of an exact sequence $X \to P \to Y$ where P is projective injective. In the Frobenius category $\mathcal{F}_\pi$, for each indecomposable nonprojective object $E(x,y)$ in $\mathcal{F}_\pi$ we choose the following exact sequence.

$$E(x,y) \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} E(y,y) \coprod E(x, x+2\pi) \xrightarrow{[1,1]} E(y, x+2\pi) \tag{3.1}$$

Here all morphisms between indecomposable objects $E(a,b)$ are diagonal (as 2×2 matrices) and labeled by the scalar $r \in R$ indicating that they are r times the basic diagonal morphism. The middle term is projective-injective. So, this choice defines the shift functor

$$T(x,y) = (y, x+2\pi)$$

in the stable category $\mathcal{C}_\pi$ of $\mathcal{F}_\pi$. (See [4] for details.) The functor T takes basic morphisms to basic morphisms.

As an example of the general construction and the precise nature of the triangulated structure of the continuous cluster category, we give the following example of a distinguished triangle in $\mathcal{C}_\pi^{(2)}$.

Example 3.3.1. Take any 6 distinct points $a < b < c < x < y < z < a + 2\pi$. Let $X = (a,x)$, $Y = (b,z) \coprod (c,y)$. Then any morphism $\underline{f} : X \to Y$, both of whose components are nonzero, can be completed to a distinguished triangle

$$X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} TX$$

if and only if $Z \cong (b, a+2\pi) \coprod (c,z) \coprod (x,y)$. Furthermore, this triangle will be distinguished if and only if the matrices $(f_i), (g_{ji}), (h_j)$ of the morphisms $\underline{f}, \underline{g}, \underline{h}$ satisfy the five conditions listed in the figure below.

(1) $h_1 g_{11} f_1 = -1$
(2) $h_2 g_{21} f_1 = 1$
(3) $h_2 g_{22} f_2 = -1$
(4) $h_3 g_{32} f_2 = 1$
(5) $g_{12} = g_{31} = 0$

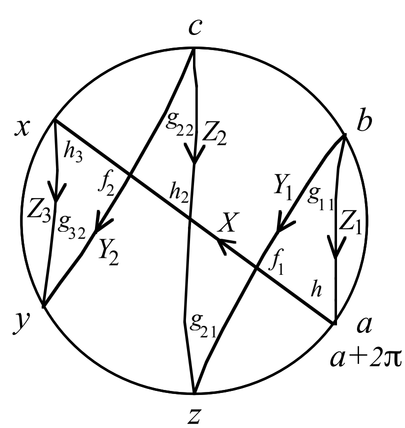

FIGURE 1. In the four triangles in the figure, the product of the angles is 1 or -1 depending on the orientation of X around its boundary. $g_{12} = 0$ since Y_2 does not meet Z_1. Similarly, $g_{31} = 0$.

This is an example of a general procedure for determining which candidate triangles are distinguished as we will explain in other papers.

Proof. By definition the distinguished triangles starting with $\underline{f} : X \to Y$ is given by lifting $\underline{f}$ to a morphism in $\mathcal{F}_\pi$ and taking the pushout of the chosen sequence for X:

$$\begin{array}{ccccc}
X = E(a,x) & \xrightarrow{\begin{bmatrix}1\\-1\end{bmatrix}} & E(x,x)\coprod E(a,a+2\pi) & \xrightarrow{[1,1]} & TX = E(x,a+2\pi) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle =} \\
Y = E(b,z)\coprod E(c,y) & \xrightarrow{g} & Z & \xrightarrow{h} & E(x,a+2\pi)
\end{array}$$

Since the bottom row is an exact sequence in $\mathcal{F}_\pi$, as an object of $\mathcal{P}_{S^1}$ we must have $Z = P_b \coprod P_c \coprod P_x \coprod P_y \coprod P_z \coprod P_{a+2\pi}$ and these objects must be paired to make 3 indecomposable objects of $\mathcal{F}_\pi$. One of them must be $E(b,w)$ for some w. But the morphism g maps the summand P_b of $E(b,z)$ isomorphically onto the summand P_b of $E(b,w)$ and this is only possible if $w = z$ or $w = a+2\pi$. The first case is not possible since $\mathcal{C}_\pi((c,y),(b,z)) = 0$ which would force $f_1 = 0$ contrary to assumption. So, $w = a+2\pi$ and $E(b,a+2\pi)$ is a summand of Z. An analogous argument shows that $E(x,y)$ is also a summand of Z and the remaining two points must be paired to give $E(c,z)$ as claimed.

Next we show that the 5 conditions are sufficient to have a distinguished triangle. To do this we first lift the elements $f_i, g_{ji}, h_j \in K$ to R so that the 5 conditions are still satisfied. Then we let u be given by the 3×2 matrix with entries in R given by:

$$u = \begin{bmatrix} 0 & g_{11}f_1 \\ 0 & 0 \\ g_{32}f_2 & 0 \end{bmatrix}$$

Then the diagram commutes and $Y \to Z \to TX$ is the pushout of the chosen sequence for X making $X \to Y \to Z \to TX$ a distinguished triangle by definition.

Finally we prove the necessity of the 5 listed conditions. The last condition follows from the fact that $\mathrm{Hom}(Y_2, Z_1) = 0 = \mathrm{Hom}(Y_1, Z_3)$. The condition $\underline{gf} = 0$ implies that $g_{21}f_1 + g_{22}f_2 = 0$. So, (2), (3) are equivalent. The condition $\underline{hg} = 0$ implies that $h_1g_{11} + h_2g_{21} = 0$ and $h_2g_{22} + h_3g_{32} = 0$. So, (1) – (4) are all equivalent. So, it suffices to prove (1).

The composition $X = (a,x) \xrightarrow{f_1} Y_1 = (b,z) \xrightarrow{g_{11}} Z_1 = (b, a+2\pi)$ factors through $(a, a+2\pi)$. Modulo the maximal ideal, the induced morphism $(a,a+2\pi) \to Z_1$ must be $-g_{11}f_1$ times the basic morphism since the chosen map $X = (a,x) \to (a,a+2\pi)$ in (3.1) is -1 times the basic map. However, the composition $(a,a+2\pi) \to Z_1 \xrightarrow{h_1} TX$ is equal to the basic map $(a,a+2\pi) \to TX$ since $Y \to Z \to TX$ is the pushout of (3.1) by definition of distinguished triangles. Therefore, $h_1(-g_{11}f_1) = 1$ proving (1). □

3.4. **Generalizations.** Some of the basic theorems in this paper hold much more generally with the same proofs.

Proposition 3.4.1. *Let $\mathcal{P}$ to be any additive Krull-Schmidt R-category where the endomorphism rings of indecomposable objects are commutative local R-algebras. Let $\mathcal{F}$ be the category of all pairs (V,d) where V is an object of $\mathcal{P}$ and d is an endomorphism of V with d^2 equal to multiplication by the uniformizer $t \in R$. Take exact sequences in $\mathcal{F}$ to be sequences $(X,d) \to (Y,d) \to (Z,d)$ where $X \to Y \to Z$ is split exact in $\mathcal{P}$. Then $\mathcal{F}$ is a Frobenius category with projective-injective objects $(X^2, \begin{bmatrix}0 & t\\ 1 & 0\end{bmatrix})$ for X in $\mathcal{P}$.* □

This implies that the stable category $\mathcal{C} = \underline{\mathcal{F}}$ becomes a triangulated category once we choose, for each object (V, d) an exact sequence:

$$(V,d) \xrightarrow{\begin{bmatrix} d \\ 1 \end{bmatrix}} \left(V^2, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right) \xrightarrow{[-1,d]} (V, -d)$$

Then $T(V,d) = (V, -d)$.

The part that fails in general is Theorem 2.2.4, the determination of all objects in $\mathcal{F}$ and thus in $\mathcal{C}$. One can also replace the condition $d^2 = t$ with $d^n = t$ for any $n \geq 2$. The analogous construction gives a Frobenius category. However, for $\mathcal{P} = \mathcal{P}_{S^1}$, $n \geq 3$ and any $k \geq 0$, this Frobenius category has an indecomposable object (V, d) so that V has $n(n-1)^k$ components (as an object of $\mathcal{P}$).

Remark 3.4.2. One interesting feature of this general construction is that, in the special case of the continuous Frobenius category $\mathcal{F}_\pi$, the chosen exact sequence above gives a different topology on the continuous cluster category $\mathcal{C}_\pi$. The space of indecomposable objects is homeomorphic to the disconnected two fold covering of the open Moebius band using this general construction as opposed to the connected (and oriented) two-fold covering which we are using.

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