# CONTINUOUS CLUSTER-TILTED CATEGORIES

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Abstract. We show that the quotients of the continuous cluster category $\mathcal{C}_\pi$ and the rational subcategory $\mathcal{X}$ modulo the additive subcategory generated by any cluster are abelian categories and we show that they are isomorphic to categories of infinite and finite length modules, respectively, over the endomorphism ring of the cluster. These theorems extend the theorems of Caldero-Chapoton-Schiffler and Buan-Marsh-Reiten for cluster categories of type $A_n$ to their continuous and countably infinite limits respective.

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## Introduction

This paper is based on a lecture given by the first author at the University of Sherbrooke with the title "Continuous cluster categories II" and was presented at an AMS sectional meeting in Iowa under the title "Continuous spaced-out cluster category." Both lectures followed lectures by the second author entitled "Continuous cluster categories" which explained the basic constructions and properties of the continuous derived category and continuous cluster categories.

Since the first paper on this subject is becoming too long, we decided to start a second paper which begins with a review, just as in the lecture. This second paper will concentrate on the interpretation of objects in the continuous cluster tilted category $\mathcal{C}_\pi/\mathcal{T}$ as modules over the Jacobian algebra of an infinite quiver with potential.

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# 1. Review

We recall the construction of the continuous cluster categories $\mathcal{C}_c$ for $0 < c \leq \pi$ and we note that for $c' < c$ the category $\mathcal{C}_{c'}$ is a quotient of $\mathcal{C}_c$. We will then concentrate on the largest cluster category $\mathcal{C}_\pi$. All of our categories will be Krull-Schmidt $K$-categories which are strictly monoidal with respect to direct sum. In other words, all objects are finite formal sums of indecomposable objects which have local endomorphism rings. We will say that the category is "strictly additive" to emphasize that this is with respect to direct sum and not tensor product. All functors will also be strictly additive so that they are uniquely determined by their restriction to the full subcategory of indecomposable objects. Also, with one exception, all triangulated functors $F$ will be *strictly triangulated* in the sense that $FT = TF$ and $F$ takes distinguished triangles to distinguished triangles. The exception is the triangulated embedding of the standard cluster category of type $A_n$ into the corresponding continuous cluster category.

1.1. **Construction of the continuous cluster categories.** Let $\mathcal{P}$ be the strictly additive $K$-category with one indecomposable object $P_x$ for every real number $x$. Morphism are given by

$$\mathrm{Hom}_{\mathcal{B}}(P_x, P_y) = \begin{cases} K & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Composition is given by multiplication.

Let $\mathcal{B}$ be the strictly additive exact category given as follows. The indecomposable objects of $\mathcal{B}$ are the points $(x, y) \in \mathbb{R}^2$. Homomorphisms are given by

$$\mathrm{Hom}_{\mathcal{B}}((x, y), (x', y')) = \begin{cases} K & \text{if } x \leq x' \text{ and } y \leq y' \\ 0 & \text{otherwise} \end{cases}$$

Composition is given by multiplication of scalars. The morphism corresponding to $1 \in K$ is called the *basic morphism.* A sequence $0 \to A \to B \to C \to 0$ in $\mathcal{B}$ is *exact* if it is split exact in each coordinate, i.e., if its image under both projection functors $\pi_1, \pi_2 : \mathcal{B} \to \mathcal{P}$ are split exact in $\mathcal{P}$. In particular, the number of summands of $B$ is equal to the number of summands in $A \oplus C$.

For any positive real number $c$ let $\mathcal{B}_c$ be the additive full subcategory generated by all $(x, y)$ where $|y - x| \geq c$. The *continuous derived category* $\mathcal{D}_c$ is defined to be full subcategory of the quotient category $\mathcal{B}/\mathcal{B}_c$ generated by the nonzero indecomposable objects of $\mathcal{B}/\mathcal{B}_c$ which are $(x, y)$ with $|y - x| < c$. Thus $\mathcal{D}_c$ has exactly one object in every isomorphism class. Since $\mathcal{B}_c$ is an approximation subcategory for $\mathcal{B}$, we have the following. (See [3].)

**Theorem 1.1.1.** *$\mathcal{D}_c$ is a triangulated category. If $X = (x, y)$ then $TX = (y + c, x + c)$.*

We use the sign convention that "positive triangles" (or "up-right-up" triangles) have positive signs on their morphisms. Thus for any $x - c < y < z < x + c$ the sequence

$$(x, y) \xrightarrow{1} (x, z) \xrightarrow{1} (y + c, z) \xrightarrow{1} (y + c, x + c),$$

with all morphisms being basic morphism, is a distinguished triangle. Also the "negative triangle" (or "right-up-right" triangle)

$$(x, y) \xrightarrow{1} (w, y) \xrightarrow{1} (w, x + c) \xrightarrow{-1} (y + c, x + c),$$

is distinguished. Up to isomorphism, these are all the distinguished triangles in $\mathcal{D}_c$ with each term being indecomposable.

Next, we create the *doubled category* $\mathcal{D}_c^{(2)}$ given by adding an additional copy $(x, y)'$ of every indecomposable object $(x, y)$ in $\mathcal{D}_c$ together with an isomorphism $\eta : (x, y)' \cong (x, y)$ with the property that $T$ anticommutes with $\eta$. In other words, $T(X') = (TX)'$ but

$$T\eta = -\eta_{TX'} : TX' \to TX$$

This implies that a negative triangle with positive signs:

$$(x,y)' \xrightarrow{1} (w,y) \xrightarrow{1} (w, x+c) \xrightarrow{1} (y+c, x+c)'$$

is a distinguished triangle since it is isomorphic to the triangle above. We say that $(x,y)'$ has the *opposite parity* as $(x,y)$ and we define $(x,y)'' = (x,y), \eta' = \eta^{-1} : (x,y) \cong (x,y)'$.

For any $d \geq c$ the continuous cluster category $\mathcal{C}_{c,d}$ is defined to be the orbit category $\mathcal{C}_{c,d} := \mathcal{D}_c^{(2)}/F_d$ where $F_d$ is the triangulated functor on the doubled category $\mathcal{D}_c^{(2)}$ defined by

$$F_d(x,y) = (y+d, x+d)'$$

Since $F_d$ takes positive triangles to negative triangles, the change in parity is necessary to make $F_d$ strictly triangular. We denote the orbit of $(x,y)$ in $\mathcal{C}_{c,d}$ by $M(x,y)$. So, $M(x,y) = M(y+d, c+d)'$.

**Theorem 1.1.2.** *The category $\mathcal{C}_{c,d}$ is triangulated so that the orbit map $\mathcal{D}_c^{(2)} \to \mathcal{C}_{c,d}$ is strictly triangulated and all distinguished triangles $A \to B \to C \to TA$ with indecomposable $A, B, C$ are images of distinguished triangles in $\mathcal{D}_c^{(2)}$.*

It is easy to see that, up to isomorphism, $\mathcal{C}_{c,d}$ depends on the ratio $c/d$ in the sense that $\mathcal{C}_{c,d} \cong \mathcal{C}_{ac,ad}$ for any positive real number $a$. Therefore, we can fix $d$ to be any convenient number. So, we choose $d = \pi = 3.14159\cdots$ and we use the notation:

$$\mathcal{C}_c := \mathcal{C}_{c,\pi}.$$

**Theorem 1.1.3.** *If $\dfrac{c}{\pi} = \dfrac{n+1}{n+3}$ then there is a triangulated embedding of the cluster category of type $A_n$ into $\mathcal{C}_c$.*

When $c < \pi$ we define two indecomposable objects $X, Y$ of $\mathcal{C}_c$ to be *compatible* if $\mathrm{Ext}^1(X,Y) = 0 = \mathrm{Ext}^1(Y,X)$. Recall that $\mathrm{Ext}^1(Y,X) := Hom(Y,TX)$ in any triangulated category. When $c = \pi$ we have another type of cluster category.

**Theorem 1.1.4.** *For any positive $n$, there is a triangulated embedding of the spaced out cluster category of type $A_n$ into $\mathcal{C}_\pi$.*

We note that in $\mathcal{C}_\pi$ we have

$$T(M(x,y)) = M(y+\pi, x+\pi) = M(x,y)'.$$

Thus $TX = X'$ for all objects $X$.

1.2. **Clusters.** Recall that two indecomposable objects $X, Y$ in $\mathcal{C}_\pi$ are *compatible* if either $\mathrm{Hom}(X,Y) = 0$ or $\mathrm{Hom}(Y,X) = 0$. If $X = M(x,y)$, the compatible compatible object are $M(a,b)$ and $M(a,b)'$ where either

1. $a \leq x$ and $b \geq y$ or
2. $a \geq x$ and $b \leq y$.

Next, we need a metric on the set of indecomposable objects of $\mathcal{C}_\pi$. We use the "taki-cab metric" which is the length of the shortest path consisting entirely of vertical and horizontal line segments. So, the distance between two objects $X, Y$ is the minimum of all real numbers of the form $|x-a| + |y-b|$ if $X = M(x,y)$ and $Y = M(a,b)$ or $Y = M(a,b)'$. An *open ball* with *radius* $\epsilon$ around a point $M(x,y)$ is the set of all objects isomorphic to $M(a,b)$ where $|x-a| + |y-b| < \epsilon$.

This defines the usual topology on the set of isomorphism classes of indecomposable objects of $\mathcal{C}_\pi$ which we denote by $\mathcal{M}$. This is the *open Moebius band*

$$\mathcal{M} \equiv \{(x,y) \in \mathbb{R}^2 \mid |y-x| < \pi\}/(x,y) \sim (y+\pi, x+\pi).$$

A *cluster* $\mathcal{T}$ is defined to be a discrete maximal pairwise compatible set of nonisomorphic indecomposable objects of $\mathcal{C}_\pi$. By *discrete* we mean that for every $M(x,y)$ in $\mathcal{T}$ has an open neighborhood that contains no other object of $\mathcal{T}$. Discreteness implies that $\mathcal{T}$ is at most countably infinite.

**Proposition 1.2.1.** *Given any object $X$ in any cluster $\mathcal{T}$ in $\mathcal{C}_\pi$ there, up to isomorphism, exactly two distinguished triangles $X \to A \to B \to TX$ and $X \to C \to D \to TX$ with $A, B, C, D$ in $\mathcal{T}$. If we delete $X$ from the cluster $\mathcal{T}$ then, up to isomorphism, there is a unique object $X^*$ in $\mathcal{C}_\pi$ so that $\mathcal{T}\backslash\{X\} \cup \{X^*\}$ is a cluster and this object $X^*$ is given by the octagon axiom:*

$$\begin{tikzcd} B \arrow[dr] & & A \arrow[ll] \\ & X \arrow[ur] \arrow[dl] & \\ C \arrow[rr] & & D \arrow[ul] \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} B \arrow[dd] & & A \arrow[dl] \\ & X^* \arrow[ul] \arrow[dr] & \\ C \arrow[ur] & & D \arrow[uu] \end{tikzcd}$$

An object $X \in \mathcal{M}$ will be called *rational* with respect to a cluster $\mathcal{T}$ if it can be obtained by a finite sequence of mutations from $\mathcal{T}$. The set of rational points is countable. So, mutation does not act transitively on the set of clusters.

Let $Homeo_+(S^1)$ denote the group of orientation preserving homoeomorphisms of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Any $\varphi \in Homeo_+(S^1)$ lifts to a homeomorphism $\tilde{\varphi}$ of $\mathbb{R}$ so that $\tilde{\varphi}(x+2\pi) = \tilde{\varphi}(x)+2\pi$ for all real $x$. Define an action of this group on the cluster category $\mathcal{C}_\pi$ by

$$\varphi M(x,y) = M(\tilde{\varphi}(x), \tilde{\varphi}(y-\pi)+\pi)$$

This is independent of the choice of $\tilde{\varphi}$ and defines a triangulated automorphism of $\mathcal{C}_\pi$. Furthermore, for any triangulated automorphism $\psi$ of $\mathcal{C}_\pi$, there is a $\varphi \in Homeo_+(S^1)$ so that $\psi(X) \cong \varphi(X)$ for all objects $X$.

**Theorem 1.2.2.** *For any two clusters $\mathcal{T}, \mathcal{T}'$ in $\mathcal{C}_\pi$ there is a $\varphi \in Homeo_+(S^1)$ so that $\varphi(\mathcal{T}) \cong \mathcal{T}'$.*

Therefore, up to isomorphism, the continuous cluster category $\mathcal{C}_\pi$ has only one cluster. When we need specific coordinates for objects we will take the *standard cluster* $\mathcal{T}_0$ which is defined to be the set of objects with coordinates

$$\left(\frac{m\pi}{2^n}, \pi + \frac{(m-1)\pi}{2^n}\right)$$

for integers $n \geq 0$ and $0 \leq m < 2^{n+1}$. We need to delete $(\pi,\pi) = (0,0)'$ to avoid repetition. The other points are all nonisomorphic. The set of points which are rational with respect to $\mathcal{T}_0$ are those points with coordinates $(a\pi/2^n, b\pi/2^n)$ for all integer $a, b, n$ with $n \geq 0$.

**Definition 1.2.3.** Let $\mathcal{X}$ be the additive subcategory of $\mathcal{C}_\pi$ generated by all objects which are rational with respect to the standard cluster $\mathcal{T}_0$. Then the quotient category $\mathcal{X}/\mathcal{T}_0$ will be called the *rational cluster-tilted category.* For any cluster $\mathcal{T}$ the quotient category $\mathcal{C}_\pi/\mathcal{T}$ will be called a *continuous cluster-tilted category.*

## 2. Cluster-tilted categories are abelian

In this section we will show that $\mathcal{X}/\mathcal{T}_0$ and $\mathcal{C}_\pi/\mathcal{T}$ are abelian categories.

2.1. **Ends.** We use the functor of "ends" to show that the continuous cluster category has infinite Rouquier dimension.

One of the simplest examples of parity is the semi-simple category *mod*-$K$. The simple model for this category has exactly one object $K^n$ in every isomorphism class. In the doubled category $\mathcal{K} = \textit{mod-}K^{(2)}$, the objects are pairs $(K^n, K^m)$ which behaves like $K^{n+m}$ in the sense that

$$\mathrm{Hom}_{\mathcal{K}}((X_0, X_1), (Y_0, Y_1)) = \mathrm{Hom}_K(X_0 \oplus X_1, Y_0 \oplus Y_1)$$

Thus, as a $K$-category, $\mathcal{K} \cong \textit{mod-}K$. However, the triangulation is different (although equivalent). $T(X_0, X_1) = (X_1, X_0)$ and on morphisms:

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Distinguished triangles are defined to be the sequences $X \to Y \to Z \to TX$ so that the sequence

$$X \xrightarrow{f} Y \to Z \to TX \xrightarrow{Tf} TY$$

is exact. Since *mod*-$K$ is semi-simple, all triangles split.

We define the *category of ends* to be the infinite direct sum of copies of this triangulated doubled category $\mathcal{K}$, one copy for each element of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$:

$$\mathcal{E} := \bigoplus_{\theta \in S^1} \mathcal{K}$$

The two copies of $K$ at the point $x \in S^1$ will be denoted $(K_x, 0), (0, K_x)$. The category of ends is a semi-simple triangulated strictly additive category.

We define a *functor of ends* $E : \mathcal{C} \to \mathcal{E}$ to be the strictly additive, strictly triangulated functor given on objects by

$$EM(x,y) = \begin{cases} (K_x, K_{y+\pi}) & \text{if } M(x,y) \text{ has positive parity} \\ (K_{y+\pi}, K_x) & \text{if } M(x,y) \text{ has negative parity} \end{cases}$$

where $K_x$ denotes $K$ supported at the point $x \in S^1$. Since this respects the identification in the cluster category: $M(x,y)_+ \simeq M(y+\pi, x+\pi)_-$, we see that $E$ is well-defined. On morphisms $E$ is trivial on $\mathrm{Hom}(M(x,y), M(x',y'))$ except in the case where $x = x'$ and/or $y = y'$ in which case it is the identity map on the respective ends. For example $E$ sends $a \in K = \mathrm{Hom}(M(x,y)_+, M(x,z)_-)$ to

$$\begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} : (K_x, K_{y+\pi}) \to (K_{z+\pi}, K_x)$$

The triangle functor $T$ changes the sign of both of these morphisms and the other possible cases are all similar. So, $E$ strictly commutes with $T$.

**Lemma 2.1.1.** *The functor $E$ is strictly triangulated and its image is the triangulated full (not) subcategory of objects $(X,Y)$ where $X$ and $Y$ have the same dimension and not more than half of $X \oplus Y$ is supported at any point in $S^1$. (Not quite! We need a restriction on the morphisms.)*

*Proof.* For any two distinct point $x, y \in S^1$, we have $(K_x, K_y) = EM(x, y-\pi)$. So an object $(X,Y)$ is in the image of $E$ if and only if $X, Y$ have the same dimension, say $n$, and $X = \oplus K_{x_i}$, $Y = \oplus K_{y_j}$ where the indices can be renumbered in such a way that $x_i \neq y_i$ for all $1 \le i \le n$. This is a perfect matching problem which by Hall's theorem can be solved if and only if not more than half of the $2n$ points $x_i, y_i$ are equal. $\square$

If we delete at least one point in $S^1$ we can ignore the restriction on objects and morphisms since we can counterbalance all the negative points by positive points at the missing point and similarly for the positive points. Let $\mathcal{K}_0$ denote the quotient category $\mathcal{K}$ modulo the object supported at 0. Then the functor of ends gives a strictly triangulated strictly additive functor from $\mathcal{C}$ onto $\mathcal{K}_0$. The computation of Rouquier dimension comes from the simple observation that the set of points in $S^1$ supporting an object $X \in \mathcal{C}$ will also support any object of $\langle X \rangle_n$ for any $n \geq 1$.

**Proposition 2.1.2.** *The triangulated categories $\mathcal{X}$ and $\mathcal{C}$ have infinite Rouquier dimension. Moreover, the triangulated category $\mathcal{C}$ cannot be constructed in any countable sequence of steps. I.e., there does not exist a sequence of objects $X_n$ in $\mathcal{C}$ so that $\mathcal{C} = \bigcup \langle X_n \rangle_n$.*

*Proof.* Any object $X$ has only a finite number of ends and the process of thickening and extending does not increase the set of ends. So all objects of $\bigcup_n \langle X \rangle_n$ will have their ends in this same finite set. Therefore, $\mathcal{X}$ and $\mathcal{C}$ cannot be generated by single elements. For $\mathcal{C}$, any countable collection of objects will have at most countably many ends. So, $\mathcal{C}$ cannot be generated by any countable sequence of objects $X_n$. □

2.2. **Mesh.**

**Definition 2.2.1.** The *mesh* of a finite subset of $\mathbb{R}/2\pi\mathbb{Z}$ is defined to be the smallest positive difference between two elements. For any objects $X$ in $\mathcal{C}$, the *mesh* of $X$ is defined to be the mesh of the set of ends of $X$.

In the next proposition we use the notation $S_\epsilon = M(a+\epsilon, b+\epsilon)$ if $S = M(a,b)$.

**Proposition 2.2.2.** *If $0 < \epsilon < \delta < \mathrm{mesh}(S \oplus X)$ then $\mathrm{Hom}_{\mathcal{C}}(X, S_\epsilon) \cong \mathrm{Hom}_{\mathcal{C}}(X, S) \cong \mathrm{Hom}_{\mathcal{C}}(X, S_\delta)$ and $\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(S_\epsilon, X) \cong \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(S_\delta, X)$.*

*Proof.* Let $X = M(x,y)$ and $S = M(a,b)$. Then we have

$$\mathrm{Hom}_{\mathcal{C}}(X,S) = \begin{cases} K & \text{if } x \leq a < y+\pi \text{ and } y \leq b < x+\pi \\ 0 & \text{if this does not hold for any choice of coordinates for } S \end{cases}$$

$$\mathrm{Hom}_{\mathcal{C}}(X,S_\epsilon) = \begin{cases} K & \text{if } x \leq a+\epsilon < y+\pi \text{ and } y \leq b+\epsilon < x+\pi \\ 0 & \text{if this does not hold for any choice of coordinates for } S \end{cases}$$

However, the condition on the mesh implies that the condition $x \leq a+\epsilon < y+\pi$ is equivalent to the condition $x \leq a < y+\pi$ and similarly for the other condition. Therefore, $\mathrm{Hom}_{\mathcal{C}}(X, S_\epsilon) = \mathrm{Hom}_{\mathcal{C}}(X,S)$ for all positive $\epsilon < \mathrm{mesh}(S \oplus X)$ and thus also for $\epsilon$ replaced with $\delta$.

For morphisms in $\mathcal{C}/\mathcal{T}$, we need to avoid $S$ and all other objects in $\mathcal{T}$. If a nonzero morphism $S_e \to X$ factors through an object $T$ in $\mathcal{T}$ with coordinates $(s,t)$ then we must have:

$$x \leq a+\epsilon \leq s < y+\pi \text{ and } y \leq b+\epsilon \leq t < x+\pi$$

□

2.3. **Support.** $\mathcal{C}$ is the continuous cluster category. $\mathcal{M}$ is the set of indecomposable objects of $\mathcal{C}$. The open subsets of $\mathcal{M}$ are the inverse images of the open subsets of the open Moebius band.

We use the notation $\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X,Y)$ is the vector space of all homomorphism $X \to Y$ in $\mathcal{C}$ modulo the subspace of those which factor through some object of $\mathcal{T}$.

The infinitesimal $\tau^{-1}S$ plays the role of the projective modules of the usual cluster tilted algebras (and $\tau S$ plays the role of the injective modules). The support of an object is analogous to the support of an object. We also need the infinitesimal analogue of the radical of the projective. Each infinitesimal radical has two components. These objects are "pro-objects" in the category. They are formal inverse limits of sequences of objects in the category.

**Definition 2.3.1.** The *infinitesimal radical* of $\tau^{-1}S$ which we denote by $r\tau^{-1}S$ is formally defined as follows.

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(r\tau^{-1}S, X) = \lim_{\epsilon\to 0+} \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(M(x+\epsilon, y) \oplus M(x, y+\epsilon), X)$$

Strictly speaking, this only defines a covariant functor on the category $\mathcal{C}/\mathcal{T}$.

**Definition 2.3.2** (infinitesimal Auslander-Reiten translation)**.** If $S(x,y) \in \mathcal{T}$ and $X$ is an object of $\mathcal{C}$, we define

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) := \lim_{\epsilon\to 0+} \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(M(x+\epsilon, y+\epsilon), X)$$

and similarly,

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X, \tau S) := \lim_{\epsilon\to 0+} \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X, M(x-\epsilon, y-\epsilon))$$

The groups $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}S, X), \mathrm{Hom}_{\mathcal{C}}(X, \tau S)$ are defined analogously with $\mathcal{C}/\mathcal{T}$ replaced by $\mathcal{C}$.

**Proposition 2.3.3** (infinitesimal Auslander-Reiten duality)**.** *There is a natural duality:*

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) \cong D\,\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X, \tau S)$$

*where* $D = \mathrm{Hom}_K(-, K)$ *is vector space duality.*

*Proof.* The duality is given by composition:

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) \otimes \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X, \tau S) \to \lim \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(M(x+\epsilon, y+\epsilon), M(x-\epsilon, y-\epsilon)) = K$$

since every term in the double limit is equal to $K$. □

**Proposition 2.3.4.** *Suppose that* $S \in \mathcal{T}$ *and* $X \in \mathcal{M}$ *and* $S, X$ *are not isomorphic. Then the following are equivalent.*

1. $\mathrm{Hom}(S, X) \neq 0$ *and* $\mathrm{Hom}(X, S) \neq 0$.
2. *There is a neighborhood* $U$ *of* $S$ *in* $\mathcal{M}$ *so that* $\mathrm{Hom}(S', X) \neq 0$ *for all* $S' \in U$.
3. *There is a neighborhood* $V$ *of* $S$ *in* $\mathcal{M}$ *so that* $\mathrm{Hom}(X, S') \neq 0$ *for all* $S' \in V$.
4. $\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) \neq 0$
5. $\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X, \tau S) \neq 0$.

The *support* of an indecomposable object $X$ is defined to be the set of all objects $S \in \mathcal{T}$ satisfying any of the equivalent conditions given above. By definition, an object cannot be in its own support. The *support* of any object $X$ in $\mathcal{C}$ is defined to be the union of the supports of its components. The *length* of an indecomposable object is defined to be the size of its support. The *length* of any object $X$ is the sum of the lengths of its components.

**Proposition 2.3.5.** *An object of* $\mathcal{C}$ *lies in* $add\,\mathcal{T}$ *if and only if its support is empty. The support of an object is finite if and only if the object lies in* $\mathcal{X}$.

*Proof.* If $\mathrm{Hom}_{\mathcal{C}}(S, T)$ and $\mathrm{Hom}_{\mathcal{C}}(T, S)$ are both nonzero and $S, T$ are nonisomorphic, then $S, T$ are not compatible. Therefore, $S, T$ cannot both lie in $\mathcal{T}$. So, the support of any object of $\mathcal{T}$ is empty.

Conversely, if $\mathrm{Hom}_{\mathcal{C}}(T, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, T)$ are both equal to zero for all $T \in \mathcal{T}$ then $X$ is compatible with every object of $\mathcal{T}$. So, $X$ lies in $add\,\mathcal{T}$.

If $X$ lies in $\mathcal{X}$ then we know that the support is finite.

Now suppose that $X = M(x_0, y_0 - \pi)$ does not lie in $\mathcal{X}$. Suppose by symmetry that $x_0$ does not have the form $a\pi/2^n$. Since $X$ is not in $\mathcal{T}$, there is at least one object $T$ in the support of $X$ with coordinates, say, $(x, y - \pi)$. Going in the direction of the boundary point $(x_0, x_0 - \pi)$, the object $T = M(x, y - \pi)$ has two children:

$$\left(x, \frac{x+y}{2} - \pi\right), \quad \left(\frac{x+y}{2}, y - \pi\right)$$

The left child lies in the support of $X = M(x_0, y_0 - \pi)$ if $\frac{x+y}{2} - \pi > x_0 - \pi$. The right child lies in the support of $X$ if $\frac{x+y}{2} < x_0$. Clearly, only one of these inequalities can hold. If neither holds then we must have $x_0 = \frac{x+y}{2}$ which means that $x_0$ has the form $x_0 = a\pi/2^n$. Since this is not the case by assumption, $T$ has exactly one child $T_1$ in the support of $X$. By repeating this, $T_1$ also has exactly one child $T_2$ in the support of $X$. Proceeding in this way, we get an infinite sequence $T_i$ of objects of $\mathcal{T}$ which lie in the support of $X$. $\square$

## 2.4. **Kernels.**

**Lemma 2.4.1.** *The rational cluster tilted category has kernels and cokernels.*

*Proof.* Since $\mathcal{X}/\mathcal{T}$ is isomorphic to its opposite category, we only need to show that morphisms have kernels.

Take any morphism $f : X \to Y$. Let $p : T^Y \to Y$ be the minimal contravariant $\mathcal{T}$ approximation of $Y$. Then the fiber is also in $\mathcal{T}$ and we have a distinguished triangle $T_1 \to T^Y \to Y \to T_1^-$. If $A$ is the fiber of the composition $X \to Y \to T_1^-$ then we have another distinguished triangle

$$T_1 \to A \xrightarrow{j} X \to T_1^-$$

Then it follows from the octagon axiom that $A$ is the pull-back in the diagram:

$$\begin{tikzcd}
T_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & T_1^- \arrow[d] \\
T_1 \arrow[r] & T^Y \arrow[r, "p"] & Y \arrow[r] & T_1^-
\end{tikzcd}$$

Equivalently, we have another distinguished triangle in $\mathcal{C}$:

$$Y^- \to A \to X \oplus T^Y \xrightarrow{(f,p)} Y$$

We claim that $A$ is the kernel of $f$ in $\mathcal{X}/\mathcal{T}$. The composition $A \to X \to Y$ is zero since it factors through $T^Y$. Suppose that $g : Z \to X$ is an morphism in $\mathcal{C}$ so that $f \circ g : Z \to Y$ factors through some object $T$ of $\mathcal{T}$. Then it factor through a map $h : Z \to T^Y$. Then $fg - ph = 0$ so we can lift to $A$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z \arrow[r, "="] \arrow[d, dashed] & Z \arrow[r] \arrow[d, "\binom{g}{-h}"'] & 0 \arrow[d] \\
Y^- \arrow[r] & A \arrow[r] & X \oplus T^Y \arrow[r, "(f,p)"] & Y
\end{tikzcd}$$

We still need to show that the morphism $Z \to A$ constucted above is unique. This is equivalent to showing that $A \to X$ is a monomorphism.

First we note that for $Z = T^X$, the contravariant approximation of $X$, the argument above shows that the mapping $T^X \to X$ factors through $A$. Take any morphism $t : Z \to A$ so that the composition $Z \xrightarrow{t} A \xrightarrow{j} X$ factors through an object of $\mathcal{T}$ which we can take to be $T^X$. Lifting the map $T^X \to X$ to $A$, we have that the difference between $t : Z \to A$ and some other mapping $s : Z \to A$ which factors through $T^X$ has the property that $j(t - s) = 0$. But this implies that the morphism $t - s : Z \to A$ factors through the fiber of $j : A \to X$. But that fiber is $T_1 \in \mathcal{T}$. Therefore, $t - s$ and thus also $t : Z \to A$ is zero in $\mathcal{X}/\mathcal{T}$. $\square$

**Lemma 2.4.2.** *$f : X \to Y$ is a monomorphism in $\mathcal{X}/\mathcal{T}$ if and only if the induced map*

$$\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, X) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, Y)$$

*is a monomorphism for all $S \in \mathcal{T}$.*

*Proof.* Let $S = M(x, y)$. If $X \to Y$ is a monomorphism then the induced map

$$\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(W, X) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(W, Y)$$

is a monomorphism for all indecomposable $W$ not $\mathcal{T}$. This is also true for $W = \tau^{-1}S$ since $M(x + \epsilon, y + \epsilon)$ is not in $\mathcal{T}$ for all sufficient small $\epsilon > 0$.

Conversely, suppose that $\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, f)$ is a monomorphism. Let $A$ be the kernel of $f : X \to Y$. Then we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, A) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, X) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, Y)$$

This implies that $\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, A) = 0$. This means $S$ is not in the support of $A$. Since this holds for all $S \in \mathcal{T}$, the support of $A$ is empty. So, $A$ lies in $\mathcal{T}$ making it equal to zero in $\mathcal{X}/\mathcal{T}$. □

**Lemma 2.4.3.** *If $f : X \to Y$ is a morphism in $\mathcal{X}$ the following are equivalent.*

(1) *$f$ is an epimorphism in $\mathcal{X}/\mathcal{T}$*
(2) *The induced mapping*

$$\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(Y, \tau S) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X, \tau S)$$

*is a monomorphism for all $S \in \mathcal{T}$.*

(3) *The induced map*

$$\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, X) \to \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(\tau^{-1}S, Y)$$

*is an epimorphism for all $S \in \mathcal{T}$.*

*Proof.* The equivalence between (1) and (2) is analogous to the previous lemma. The equivalence between (2) and (3) follows from infinitesimal AR-duality. □

For any $S \in \mathcal{T}$ and $X \in \mathcal{C}$ let $\mathrm{Hom}_0(S, X)$ denote the quotient of $\mathrm{Hom}_{\mathcal{C}}(S, X)$ by all morphisms which factor through an object $T \in \mathcal{T}$ which is not isomorphic to $S$. Then one can easily see that

**Lemma 2.4.4.** $\mathrm{Hom}_0(S, X) \cong \mathrm{Hom}_0(S, T_0^X) \cong K^n$ *where $n$ is the number of times that $S$ occurs as a direct summand of $T_0^X$.*

**Lemma 2.4.5.** *For any $S \in \mathcal{T}$ and object $X$ of $\mathcal{C}$ is a natural four term exact sequence*

$$0 \to \mathrm{Hom}_0(S, T_1^X) \to \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) \to \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(r\tau^{-1}S, X) \to \mathrm{Hom}_0(S, X) \to 0$$

**Theorem 2.4.6.** *The categories $\mathcal{X}/\mathcal{T}$ and $\mathcal{C}/\mathcal{T}$ are abelian.*

*Proof.* It suffices to show that every morphism which is both a monomorphism and an epimorphism is an isomorphism. So, suppose that $f : X \to Y$ is such a morphism and consider the 4-term sequence above we get:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_0(S, T_1^X) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, X) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(r\tau^{-1}S, X) \arrow[r] \arrow[d, "\gamma"] & \mathrm{Hom}_0(S, X) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_0(S, T_1^Y) \arrow[r] & \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(\tau^{-1}S, Y) \arrow[r] & \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(r\tau^{-1}S, Y) \arrow[r] & \mathrm{Hom}_0(S, Y) \arrow[r] & 0
\end{tikzcd}$$

By the lemma above $f$ induces an isomorphism on supports. So, $\beta$ is an isomorphism. Since $r\tau^{-1}S$ represents a sequence of objects which do not lie in $\mathcal{T}$, we have that $\gamma$ is a monomorphism Chasing the diagram we conclude that $\alpha$ is an isomorphism, so that $T_1^X \cong T_1^Y$ and $\delta$ is a monomomorphism, so that $T_0^X$ is a direct summand of $T_0^Y$. Call the other summand $T_2$. Then the morphism $f$ fits into the following map of triangles:

$$\begin{tikzcd}
T_1^X \arrow[r] \arrow[d, "\cong"] & T_0^X \arrow[r] \arrow[d, "j"] & X \arrow[r] \arrow[d, "f"] & T_1^X \arrow[d, "\cong"] \\
T_1^Y \arrow[r] & T_0^Y \arrow[r] & Y \arrow[r] & T_1^Y
\end{tikzcd}$$

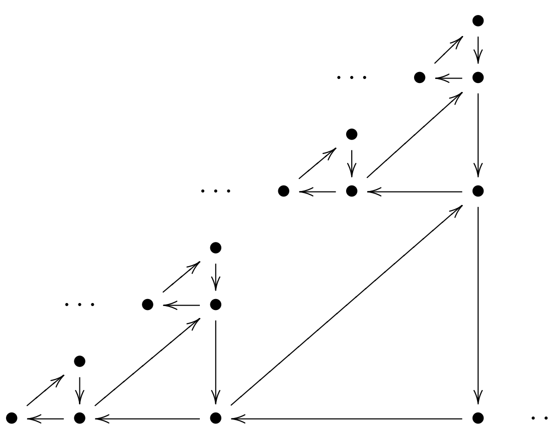
Since $T_1^X \cong T_1^Y$, the middle square is homotopy cartesian. So, $f$ completes to a triangle

$$X \to Y \to T_2 \to X$$

But this splits, with splitting given by $T_2 \to T_0^Y \to Y$. So, $Y \cong X \otimes T_2$ where $T_2 \in add\,\mathcal{T}$. Therefore $f : X \to Y$ is an isomorphism in $\mathcal{C}/\mathcal{T}$ as claimed. □

## 3. The infinite Jacobian algebra

Let $\Lambda$ be the infinite algebra without unit defined to be the subring of $\mathrm{End}_{\mathcal{X}}(T)$ consisting of all endomorphism which are zero on all but finitely many components of $T$ where $T$ is the direct sum of indecomposable objects of $\mathcal{T}$, choosing one object from each isomorphism class. For example, we can take all objects to have positive parity in some fixed fundamental domain. This is the Jacobian algebra of an infinite quiver with potential given by taking the unique infinite simply connected planar trivalent tree and replacing each vertex with a triangle and each edge with a shared vertex of two triangles. Orient all the triangles clockwise and take the potential to be the sum of all of these 3-cycles. Then the Jacobian algebra $\Lambda$ is given by the infinite quiver $Q$ with the relation that the composition of any two arrows in the same triangle is zero.



### 3.1. **Finitely generated modules over the Jacobian.**

**Proposition 3.1.1.** *Every finitely generated modules over $\Lambda$ is a direct sum of finitely many string modules which may be infinite dimensional.*

*Proof.* Choose a subquiver $Q'$ of $Q$ which is a connected union of triangles containing the supports of the generators of a f.g. module $M$. Let $I'$ be the ideal in $KQ'$ generated by the relations in $Q$ which are supported on $Q'$. Then $KQ'/I'$ is a string algebra. So, the restriction $M' = M|Q'$ of $M$ to $Q'$ is a direct sum of finitely many string modules. We will say that a vertex of $Q'$ is a *boundary point* if it lies in only one triangle of $Q'$. Let $Q''$ be obtained from $Q'$ by adding one triangle $u \to v \to w \to u$ to each boundary point $v$ of $Q'$. Let $M'' = M|Q''$.

Claim 1. For each triangle $u \to v \to w \to u$ in $Q''$ with only one vertex $v$ in $Q'$, we have $M_u = 0$ and the linear map $M_v \to M_w$ is an epimorphism.

Proof: $M$ is generated by elements of $KQ'$ but there are no paths from $Q'$ to $u$. Also, any element in the cokernel of $M_v \to M_w$ cannot be in the image in $M$ of a projective module with top in $Q'$. This proves the claim.

Let $d$ be the sum of the ranks of the maps $M_v \to M_w$ from the boundary points of $Q'$ to boundary points of $Q''$. Choose the pair $Q', Q''$ so that $d$ is minimal. Note that $d$ is the sum of the dimensions of $M_w$ over all boundary points $w$ of $Q''$. Let $Q'''$ be the subquiver of $Q$ obtained by adding triangles to the boundary points of $Q''$.

Claim 2. Let $u \to v \to w \to u$ be a triangle in $Q'''$ where $v$ is a boundary point of $Q''$. Then $M_v \to M_w$ is an isomorphism.

Proof: By Claim 1, the map $M_v \to M_w$ is surjective and $M_u = 0$. So, the sum of the dimensions of $M_w$ is $\leq d$. My minimality of $d$ we must have equality and the map must be an isomorphism as claimed.

Now take any decomposition of $M''$ into string modules. This gives a decomposition of $M_v$ into one dimensional subspaces for every boundary point $v$ of $Q''$. By Claim 2 we have isomorphisms $M_v \cong M_w$ which we use to give a compatible decomposition of $M_w$ into one dimensional summands. By repeating this process for larger and larger subquivers we obtain a decomposition of the infinite module $M$ into a direct sum of string modules as claimed. □

As observed in Claim 2 in the proof above, the infinite tails of finitely generated string modules over $\Lambda$ must be eventually oriented outward (towards the infinite end).

Let *mod*-$\Lambda$ be the category of finitely generated modules over $\Lambda$ and let $f\ell mod$-$\Lambda$ denote the full subcategory of modules of finite length. Then $f\ell mod$-$\Lambda \cong f\ell mod$-$\hat{\Lambda}$ where $\hat{\Lambda} = \lim \Lambda/r^n\Lambda$ is the *completion* of $\Lambda$. For any $\Lambda$-module $M$ let $FM$ be given by

$$FM = \lim_{\to} \operatorname{Hom}_\Lambda(\Lambda/r^n\Lambda, M)$$

Viewing $\Lambda/r^n\Lambda$ as an $\hat{\Lambda}$-$\Lambda$-bimodule we see that this gives a functor $F : mod\text{-}\Lambda \to mod\text{-}\hat{\Lambda}$.

**Lemma 3.1.2.** *$F$ is a left exact functor which vanishes on projective modules. Furthermore $FM$ has finite length for all f.g. $M$ and $FW = W$ for all modules $W$ of finite length.*

*Proof.* Since Hom is left exact and direct limit is exact, it follows that $F$ is left exact. Since a projective module $P$ contains no nonzero submodule of finite length we have $\operatorname{Hom}(\Lambda/r^n\Lambda, P) = 0$ for all $n$. So $FP = 0$. Also, the last claim: $FW = W$ for modules of finite length is clear.

Now suppose that $M$ is finitely generated and not projective. Then $M$ has 0,1 or 2 infinite ends and each infinite end is an infinite directed path starting at a vertices $x_i$. Since $M$ is not projective, these vertices $x_i$ must be distinct. Let $R(x_i)$ be the infinite string modules supported on these infinite paths. Then we have a short exact sequence

$$0 \to W \to M \to \oplus R(x_i) \to 0$$

for some module $W$ of finite length. Since $\operatorname{Hom}_\Lambda(\Lambda/r^n\Lambda, R(x_i)) = 0$ for all $n$, we get by left exactness of $F$ that $FM \cong FW$. But $FW = W$. So we are done. □

This lemma shows that $F$ induces a functor on the stable category

$$\underline{F} : \underline{mod\text{-}\Lambda} \to f\ell mod\text{-}\hat{\Lambda} = f\ell mod\text{-}\Lambda$$

which can be viewed as a retraction since it has a section $f\ell mod$-$\Lambda \hookrightarrow \underline{mod\text{-}\Lambda}$. However, $\underline{F}$ is not an isomorphism since there is no stable morphism $M \to FM$ in general. However, we have the following observation.

**Proposition 3.1.3.** *Let $\mathcal{X}$ be the full subcategory of the stable category $\underline{mod\text{-}\Lambda}$ generated by all string modules with two infinite ends. Then $\underline{F}$ induces an isomorphism $\mathcal{X} \cong f\ell mod\text{-}\Lambda$.*

*Proof.* There is an inverse functor $G : f\ell mod\text{-}\Lambda \to \mathcal{X}$ which is given by sending each string module of finite length $W$ to the unique string module $M$ having two infinite ends so that $FM = W$. The key point is that

$$\underline{\mathrm{Hom}}(M_1, M_2) \cong \mathrm{Hom}(W_1, W_2)$$

which is verified by examining all possible ways that the stings might intersect. The intersection is again a string module since $Q$ is simply connected (when we fill in the triangles). □

**Theorem 3.1.4.** *The abelian category $\mathcal{X}/\mathcal{T}$ is isomorphic to $f\ell mod\text{-}\Lambda$.*

*Proof.* Claim 1. There is a bijection between the indecomposable objects of the two categories.

Take any string module $W$ of finite length $n$. Then $W$ has support $v_1, \cdots, v_n$. We have the corresponding string module $GW = M$ which has two infinite ends starting at two distinct vertices $a, b$. These are two sources in the support of $M$. The support of $W$ gives a sequence of vertices connecting $a$ and $b$ by a zig-zag path:

$$\infty\text{-end} \leftarrow a \to v_1 - v_2 \cdots v_n \leftarrow b \to \infty\text{-end}$$

These correspond to objects $T_a, T_{v_i}, T_b$ of $\mathcal{T}$ which map nontrivially to each other in the opposite direction:

$$T_a \leftarrow T_{v_1} \cdots T_{v_n} \to T_b \tag{3.1}$$

Since $Q$ is simply connected, $\mathrm{Hom}(T_a, T_b) = 0 = \mathrm{Hom}(T_b, T_a)$. In terms of coordinates, $T_a = (a_1, a_2), T_b = (b_1, b_2)$ where by symmetry we may assume $a_1 < b_1, a_2 > b_2$.

Claim 2. There are no other objects of $\mathcal{T}$ in the rectangle $[a_1, b_1] \times [b_2, a_2]$ except for the objects $T_a, T_b$ and $T_{v_i}$.

The reason is that each arrow in the diagram (3.1) is irreducible and we know that all of the objects of $\mathcal{T}$ in the rectangle $[a_1, b_1] \times [b_2, a_2]$ are connected by a sequence of irreducible maps. These represent two walks in the quiver $Q$ from $a$ to $b$ which do not go through any zero relation. But such a walk is uniquely determined by its end points. So, they are equal. This proves Claim 2.

The indecomposable object of $\mathcal{X}/\mathcal{T}$ associated to $W$ is $X = M(b_1, a_2)$. This is the universal object of $\mathcal{X}$ to which all of the objects $T_a, T_b, T_{v_i}$ map nontrivially.

Conversely, take any indecomposable object $X = M(x, y)$ in $\mathcal{X}$ which is not in $\mathcal{T}$. Let $a_2 = y$, $b_1 = x$ and let $a_1$ be the largest real number $< x$ so that $M(a_1, a_2) \in \mathcal{T}$ and let $b_1$ be the largest real number $< y$ so that $M(b_1, b_2) \in \mathcal{T}$. Then the objects of $\mathcal{T}$ in the rectangle $[a_1, b_1] \times [b_2, a_2]$ excluding the corners $T_a = M(a_1, a_2)$ and $T_b = M(b_1, b_2)$ form the support of a string module $W$ of finite length which will give back $X$ by the above construction. Thus this describes the inverse process and gives the desired bijection concluding the proof of Claim 1.

Define a *standard string module* to be one for which, at each vertex in the support we have $K$ and for each arrow we have the identity map $K \to K$. A *basic morphism* between string modules is one which is the identity map on $K$ on each vertex in the intersection of supports. Since $Q$ is simply connected, any nonzero morphism between standard string modules is a scalar multiple of a basic morphism and any composition of basic morphisms $X \to Y, Y \to Z$ is either a basic morphism $X \to Z$ if there is one or zero is there is no basic morphism $X \to Z$.

Claim 3. Suppose that $W_1, W_2$ are standard string modules of finite length and $X_1, X_2$ are the corresponding objects of $\mathcal{X}/\mathcal{T}$. Then $\mathrm{Hom}_\Lambda(W_1, W_2) = \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X_1, X_2)$.

Since each side is at most one dimensional it suffices to show that a nonzero morphism on one side implies the existence of a nonzero morphism on the other. Suppose that $\mathrm{Hom}_\Lambda(W_1, W_2) = K$. Then the supports of $W_1, W_2$ must intersect in another string module $W_3$. Let $v, v'$ be the endpoints of the support of $W_3$. Take $v$. Either $v$ is an endpoint of the support of $W_1$ or there is another vertex $w$ in the support of $W_1$ and an arrow $v \to w$. This gives an irreducible map $T_w \to T_v$ in $\mathcal{T}$. In $M_2$ there must be a vertex $a$ and an arrow $a \to v$ giving an irreducible map $T_v \to T_a$. This means that

$$T_w \to T_v \to T_a \to T'_w$$

is a distinguished triangle where $T'_w = T_w[1]$ is $T_w$ with the opposite parity. By symmetry we may assume that this is a negative triangle so that $T_w, T_v$ have the same $y$-coordinate and $T_v, T_a$ have the same $x$-coordinate. Then all of the other vertices in the part of the support of $W_1$ in the complement of the support of $W_3$ and containing $w$ will lie to the north-west of $w$. Therefore, they lie to the south-west of $a$ and that makes the top of the rectangle for $X_1$ lie below the top of the rectangle for $X_2$ but above the point $v$. If $v$ is an endpoint of $W_1$ but not of $W_2$ then the point $a$ is the new source of $M_1$, assuming by symmetry that $T_v, T_a$ have the same $x$ coordinate, we conclude that the top of the rectangle for $X_1$ is at the $y$-coordinate of the point $a$ which is at or below the top of the rectangle for $X_2$.

A similar argument at the other endpoint of $W_3$ tells us that the right side of the rectangle for $X_1$ lies to the left or is equal to the right side of the rectangle for $X_2$. Therefore, $X_1$ lies inside the rectangle for $X_2$ and lies in the upper right side of the zig-zag created by the objects of $\mathcal{T}$ going from lower right to upper left in the $X_2$-rectangle. Therefore, $\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X_1, X_2) = K$ since $X_1$ is below and to the left of $X_2$ and there are no objects of $\mathcal{T}$ between $X_1$ and $X_2$.

Finally, suppose that $\mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X_1, X_2) = K$. Then $X_1$ must be below and to the left of $X_2$. It must also be above and to the right of the zig-zag of objects of $\mathcal{T}$ going from the lower right to upper left in the $X_2$ rectangle since these are all the points which map to $X_2$ in the category $\mathcal{X}/\mathcal{T}$. The construction of the string module $W_1$ consists of going to the left and down from $X_1$ to give another rectangle. The horizontal line through $X_1$ either meets an object of $\mathcal{T}$ in the $X_2$ rectangle or it goes to an object $T_v$ outside and to the left of this rectangle. But in that case, at the object $T_w$ where the two zig-zags meet, we have a morphism into $T_w$ from the support of $X_1$ which is an arrow away from $w$ in the support of $W_1$ which means that $W_1$ maps to $W_2$. (We need to do the same analysis at the other end to confirm this.) Therefore, $\mathrm{Hom}_\Lambda(W_1, W_2) = K$ as claimed. This proves Claim 3. The theorem follows since composition of basic morphisms is given by the same rule in both categories. (The composition of basic morphisms is a basic morphism if one exists.) □

3.2. **Infinitely generated modules.** The next theorem identifies the abelian category $\mathcal{C}/\mathcal{T}$ as a category of infinitely generated modules over $\Lambda$. Let $Mod$-$\Lambda$ denote the category of all locally finite $\Lambda$-modules. These are the $\Lambda$-modules $M$ so that $M_v$ is finite dimensional for all vertices $v \in Q_0$.

Let $Rep_+\Lambda$ denote the additive full subcategory of $Mod$-$\Lambda$ generated by all string module $M$ having two infinite ends so that in each end, the arrows are either eventually all pointing outward (as when $M$ is projective) or have an infinite number of arrows going inward and outward. The disallowed ends, the ones where the arrows are eventually all pointing inward, will be called "injective ends" or ends of "injective type."

Let $Rep_0\Lambda$ denote the additive full subcategory of $Mod$-$\Lambda$ category of string modules of $\Lambda$ with either zero, one or two infinite ends so that each infinite end has an infinite number of arrows in each direction. In other words, the direction of the arrows keeps switching back and forth.

For such a string module $W$ in $Rep_0\Lambda$ let $GW = M$ be the module with two infinite ends where we do the previous construction on any finite end. Thus, at each finite end $v$ we attach the unique vertex $w$ so that the triangle containing $v$ and $w$ meets the support of $W$ only at $v$. Then we add the ray starting at $w$ and going the other way to get $M$. As before, this has the property that

$$FM = \varinjlim \mathrm{Hom}_\Lambda(\Lambda/r^n\Lambda, M) = W$$

**Proposition 3.2.1.** *A string module $M$ has the form $M = GW$ for some $W \in Rep_0\Lambda$ if and only if $M$ is not projective and if $M$ has two infinite ends which are not of injective type.* $\square$

Let $v$ be a vertex in the support of $M = GW$ and suppose that $T_v = M(a,b)$ is the corresponding object of $\mathcal{T}$. Suppose by symmetry that $b \geq a$. Also, suppose that $b - a \geq 0$ is minimal among all the vertices in the support of $M$. Then $a \leq b < a + \pi$ and

$$\theta = a + \pi - b = \frac{\pi}{2^n}$$

for some nonnegative integer $n$. Furthermore, $a$ and $b$ are both integer multiples of $\theta$. The cluster $\mathcal{T}$ contains two other points $T_{v0} = M(a - \theta/2, b)$ and $T_{v1} = M(a, b + \theta/2)$ and there are irreducible morphisms $T_{v0} \to T_v \to T_{v1} \to T'_{v0}$ forming a distinguished triangle. Thus the vertices $v0, v, v1$ form a triangle in $Q$ with arrows going the other way: $v0 \leftarrow v \leftarrow v1 \leftarrow v0$.

Since $M$ is a string module containing $v$ in its support and having two infinite ends, the support of $M$ contains exactly one of $v0, v1$. Continuing in the same direction, we have a sequence of 0's and 1's:

$$vd_1d_2d_3\cdots$$

where $d_2 = 0$ if the next vertex $vd_10$ points away from $vd_1$.

**Lemma 3.2.2.** *The object in $\mathcal{T}$ corresponding to $vd_1d_2\cdots d_m$ has coordinates $a_m, b_m$ where*

$$b_m = b + \sum_{i=1}^{m} d_i\theta/2^i, \quad a_m = b_m - \pi + \theta/2^m$$

Note that in the limit as $m \to \infty$ we get that $b_m$ monotonically increases to $b_\infty$ and $a_m$ monotonically decreases to $a_\infty = b_\infty - \pi$. There is one kind of sequence which is excluded: There cannot be an integer $N$ so that $d_i = 1$ for all $i \geq N$ because this would correspond to the case when the arrows all point inward after that point. As a consequence, the sequence $d_1, d_2, \cdots$ is uniquely determined by $b_\infty$ (and $b$ and $\theta$) and $b \leq b_\infty < b + \theta$.

If we go in the other direction, $a$ will become larger than $b$ by choice of the vertex $v$. There are three cases to consider.

1. $a = b$.
2. $a < b$ and $a$ increases in the second tail ($a_{-1} > a, b_{-1} = b$).
3. $a < b$ and $b$ decreases in the second tail ($a_{-1} = a, b_{-1} < b$).

<u>Case 1</u>. $a = b$. In the fundamental domain, this is the point $v = (0,0)$. We have $\theta = a + \pi - b = \pi$. All points $(x,y)$ in $\mathcal{T}$ with $x \leq y$ are in the second quadrant up to isomorphism with $-\pi < x \leq 0 \leq y < \pi$. The first tail lies here with $0 \leq b_\infty < \pi$. The second tail must be in the fourth quadrant with $0 \leq x < \pi$ and $0 \geq y > -\pi$. In Case 1 we have symmetry between $a$ and $b$ and in the vertices in the second tail the roles of $a, b$ are switched. We get a sequence of 0's and 1's:

$$v_{-m} = ve_1e_2e_3\cdots e_m = (a_{-m}, b_{-m})$$

where

$$a_{-m} = -\sum_{i=1}^{m} e_i\pi/2^i, \quad b_{-m} = a_{-m} - \pi + \pi/2^m$$

As before, the binary digits $e_i$ are not allowed to all become equal to one after any point. Thus the $e_i$ are uniquely determined by the limits $a_{-\infty}$ and $b_{-\infty} = a_{-\infty} - \pi$ and

$$-\pi \leq b_{-\infty} < 0 \leq b_\infty < \pi, \quad 0 \leq a_{-\infty} < \pi \leq a_\infty < 2\pi$$

<u>Case 2</u>. $a < b$ and, if we go one step in the other direction, $a$ becomes greater than $b$ and $b_{-1} = b$. Since $\theta < \pi$, either $(a + \theta, b)$ or $(a, b - \theta)$ lies in $\mathcal{T}$ and lies in the second quadrant. In Case 2, it must be the latter and the closest point on the line $y = b$ to the right of $(a,b)$ is the point $(b + \pi - \theta, b)$.

This is equivalent to the point $(a-\theta, b-\theta)$ in the fundamental domain of $S$ and this translation of the second tail is in the triangle $\theta$ units below and to the left of the triangle which contains all possible locations for the first tail. Since $S^{-1}(x,y) = (y-\pi, x-\pi)$, the coordinates of the limit point in this second triangular region are $(b_{-\infty}-\pi, a_{-\infty}-\pi)$. Both of these triangular regions are

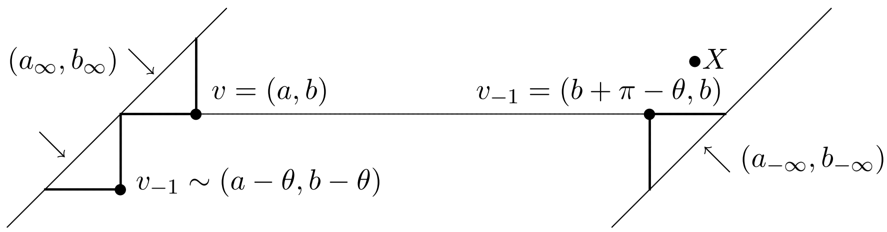



FIGURE 1. Case 2: One tail is in the triangular region with corner $(a,b)$. The other tail is in the triangular region with corner $(b+\pi-\theta, b)$ which is equivalent to $(a-\theta, b-\theta)$. $X$ is the corresponding object of $\mathcal{C}/\mathcal{T}$.

in the second quadrant. Thus, in Case 2 we have:

$$0 \le b-\theta \le b_{-\infty} < b \le b_\infty < b+\theta \le \pi$$

$$a-\theta \le a_\infty < a < b+\pi-\theta \le a_{-\infty} < b+\pi$$

Case 3. This is the same as Case 2 with the two tails reversed.

There is also Case 4 and Case 5 in the fourth quadrant which are similar to Cases 2 and 3 with the $x,y$-coordinates reversed.

**Proposition 3.2.3.** *The objects corresponding to the vertices in one tail of $GW = M$ converge to an element of $\mathbb{R}/2\pi\mathbb{Z}$ of the form $a\pi/2^m$ if and only if the tail is eventually outward pointing.*

*Remark* 3.2.4. Near the line $y = x+\pi$ (the diagonal line on the left in Figure 1), these tails correspond to horizontal lines. Near the line $y = x-\pi$ (the diagonal line on the right in Figure 1), these outwardly pointing tails are vertical lines. Note that the vertical line below $v_{-1} = (b+\pi-\theta, b)$ is equal to the horizontal line to the left of $v_{-1} = (a-\theta, b-\theta)$ in Figure 1.

*Proof.* This follows from the fact that we have excluded the possibility of sequences ending in an infinite number of 1's since these would correspond to an infinite sequence of inwardly pointing arrows. Thus every binary sequence determines a unique real number which has the form $a\pi/2^m$ if and only if it has only finitely many 1's and the rest 0's which correspond to an infinite sequence of outwardly pointing arrows. □

**Theorem 3.2.5.** *$Rep_0\Lambda$ is isomorphic to $\mathcal{C}/\mathcal{T}$.*

The argument is analogous to the finite case.

**Lemma 3.2.6.** *There is a bijection between the string modules in $Rep_0\Lambda$ and the isomorphism classes of indecomposable objects of $\mathcal{C}/\mathcal{T}$.*

*Proof.* Take any object $W$ of $Rep_0\Lambda$. Form $M = GW$ as described above. The the corresponding object of $\mathcal{C}/\mathcal{T}$ is the point $X$ with coordinates $(a_{-\infty}, b_\infty)$. The vertices in the support of $M$ form an infinite zig-zag starting at the right at the line $y = x-\pi$ and going left and up to the line $y = x+\pi$. The limiting horizontal line is $y = b_\infty$ at the top of the zig-zag and the limiting vertical line is $x = a_{-\infty}$ at the right of the zig-zag. In the case when $W$ has finite length, these form the top and right side of the rectangle $[a_1, b_1] \times [b_2, a_2]$ in the proof of Theorem 3.1.4.

Going case-by-case, the objects $M = GW$ of Case 1 correspond to the nonzero objects $X = M(x, y)$ of $\mathcal{C}/\mathcal{T}$ where $0 \leq x, y < \pi$. These are the points with the property that $\mathrm{Hom}(M(0,0), X) \neq 0$. Then $y = b_\infty \in [0, \pi)$ and $x = a_{-\infty}$ is also in the half-open interval $[0, \pi)$. In Cases 2 and 3, $X = M(x, y) = M(a_{-\infty}, b_\infty)$ lies in the second quadrant. In Cases 4 and 5, $X$ lies in the fourth quadrant.

The disallowance of injective tails gives a bijection between possible ends and the points on the unit circle. Pairs of distinct points $a, b$ on the unit circle correspond bijectively to the objects $M(a, b + \pi)$ of $\mathcal{C}$. And the object lies in $\mathcal{T}$ if and only if the corresponding string module is projective. By Proposition 3.2.1 projective modules are the only modules $M$ without injective tails which cannot occur as $GW$ for any object $W$ of finite length. Thus we have the desired bijection. □

*Proof of Theorem 3.2.5.* As in the proof of Theorem 3.1.4, it suffices to show that

$$\underline{\mathrm{Hom}}(M_1, M_2) = \mathrm{Hom}(W_1, W_2) = \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X_1, X_2)$$

where $X_1, X_2$ are the objects of $\mathcal{C}/\mathcal{T}$ corresponding to $M_1, M_2$. We can do this by reducing to the case when $W_i$ have finite length.

For each $W_i$ take a sequence of submodules $W_{in}$ of finite length so that $W_{in}$ containing any finite end of $W_i$ and also contains $n$ sinks in each infinite tail starting at some fixed vertex in the support of $W_i$. Let $X_{in}$ be the corresponding objects of $\mathcal{X}/\mathcal{T}$. Then, by Theorem 3.1.4 we have

$$\mathrm{Hom}(W_{1n}, W_{2m}) = \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X_{1n}, X_{2m})$$

Taking the limit as $m \to \infty$ we get

$$\mathrm{Hom}(W_{1n}, W_2) =_{(1)} \varinjlim \mathrm{Hom}(W_{1n}, W_{2m}) = \varinjlim \mathrm{Hom}_{\mathcal{X}/\mathcal{T}}(X_{1n}, X_{2m}) =_{(2)} \mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X_{1n}, X_2)$$

where (1) is clear and (2) follows from the fact that homorphisms vanish in a limit only when the limit line contains an object of $\mathcal{T}$ or if it contains $TX_{1n}$ at the outer edge of the domain of the functor $\mathrm{Hom}_{\mathcal{C}}(X_{1n}, -)$. But this cannot happen since $X_{1n}$ has coordinates of the form $a\pi/2^k$ and limiting lines have coordinated with $\pi$ times a number with infinite binary expansion.

Now take the limit as $n \to \infty$, for which there is no problem since there are only finitely many objects of $\mathcal{T}$ which can lie between $X_{1n}$ and $X_2$ at the point when $\mathrm{Hom}_{\mathcal{C}/\mathcal{T}}(X_{1n}, X_2) \neq 0$. □

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