Continuous Spaced-Out Cluster Category

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$$\mathbf{C}_\pi$$

井草潔 ΓT

The continuous derived category

- D_c is a triangulated category with indecomposable objects the points (x, y) in the plane R^2 so that
$$x - c < y < x + c$$
- $Hom((x, y), (a, b)) = K$ if $x \leq a < y + c$ and $y \leq a < x + c$,
$= 0$ if not.

Support of $Hom((x, y), -)$

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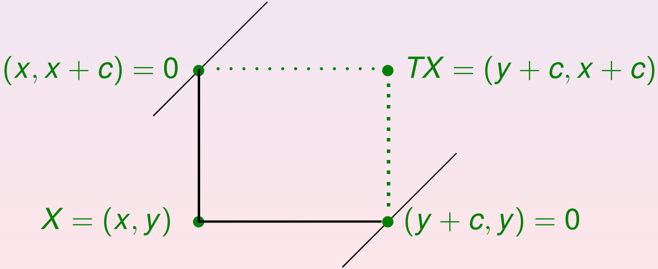


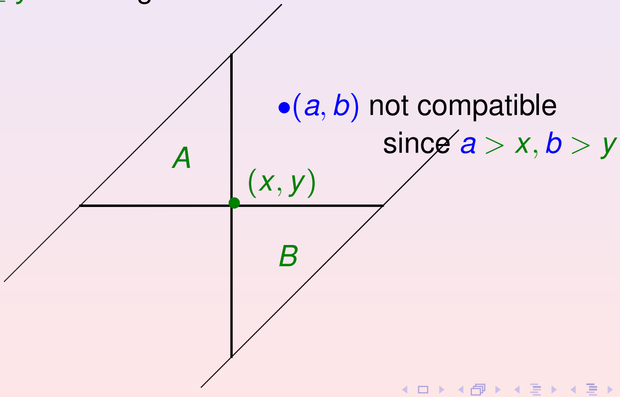
Support of $Hom((x, y), -)$

Continuous cluster categories

- C_c is the orbit category D_c/F
$$F(x,y) = (y+\pi, x+\pi)'$$
- When $\frac{c}{\pi} = \frac{n+1}{n+3}$ there is a triangulated embedding of the cluster category of type A_n:
$$C(A_n) \subseteq C_c$$
- When $c = \pi$ there is a triangulated embedding of the "spaced-out cluster category" S_n into C_π for every n.

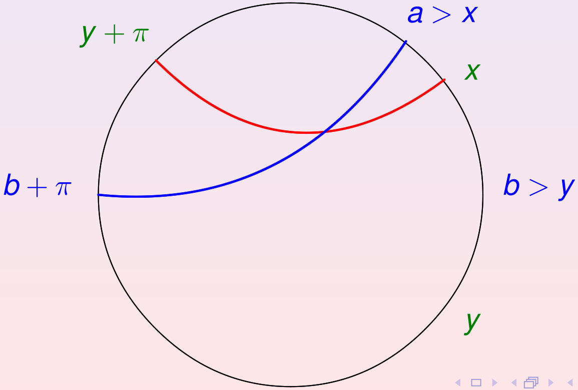
Compatibility

- (x, y) is compatible with (a, b) if either
  $Hom((a, b), (x, y)) = 0$ or $Hom((x, y), (a, b)) = 0$
- Equivalently, either $a \leq x$ and $b \geq y$ as in region A or $a \geq x$ and $b \leq y$ as in region B.



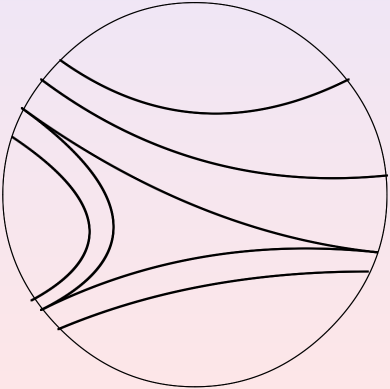
Geodesics in hyperbolic plane

- Two indecomposable objects in C_π are compatible iff the corresponding geodesics in the hyperbolic plane do not cross.
- $(x, y) \leftrightarrow$ geodesic from x to $y + \pi$ on the circle



Laminations

- A maximal collection of pairwise compatible objects is a lamination.
- This corresponds to a closed family of disjoint geodesics:



Cluster

Definition

A cluster is a discrete lamination. Discrete means every point is isolated.

Theorem

For any two clusters T, T' there exists a continuous triangulated automorphism Φ of C_π so that $\Phi(T) = T'$.

Up to isomorphism, there is only one cluster.

Cluster mutation

Proposition

Any X in any cluster T lies in exactly two distinguished triangles

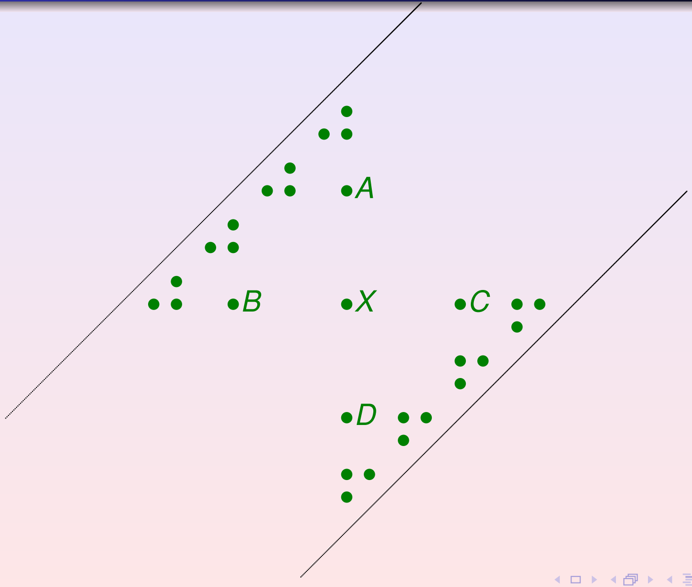
$$X \to A \to B \to TX, \qquad X \to C \to D \to TX$$

with A, B, C, D in T.

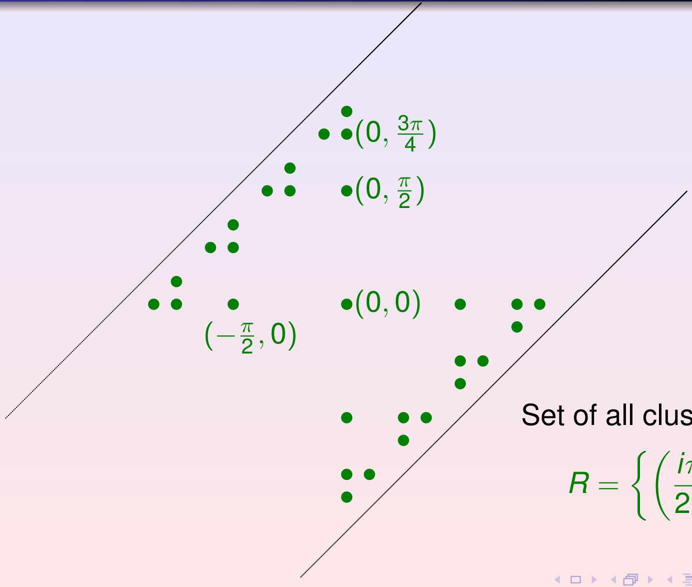
Mutation $X \leftrightarrow X^*$ is given by the octagon axiom:

$$\begin{array}{ccccc} B & & \longleftarrow & & A \\ & \searrow & & \nearrow & \\ & & X & & \\ & \swarrow & & \nwarrow & \\ C & & \longrightarrow & & D \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccc} B & & & & A \\ & \nwarrow & & \swarrow & \\ \downarrow & & X^* & & \uparrow \\ & \nearrow & & \searrow & \\ C & & & & D \end{array}$$

The standard cluster



Standard cluster: coordinates of points



Set of all cluster variables

$$R = \left\{ \left(\frac{i\pi}{2^n}, \frac{j\pi}{2^n} \right) \right\}$$

Cluster tilted categories

Definition

Let R be the full subcategory of all objects which can be obtained from the standard cluster T by mutation. The quotient categories R/T and C/T are the rational and continuous cluster tilted categories.

Theorem

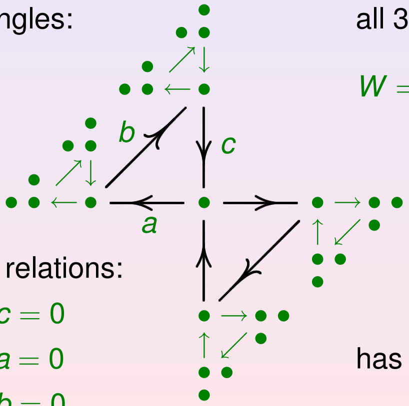
R/T and C/T are abelian categories.

Theorem

R/T and C/T are isomorphic to categories of string modules over an infinite dimensional Jacobian algebra $J(Q, W)$.

Quiver with potential

Q is an infinite union of triangles:



W is the sum of all 3-cycles

$$W = abc + \cdots$$

$J(Q, W)$ has relations:

$$\partial_a W = bc = 0$$
$$\partial_b W = ca = 0$$
$$\partial_c W = ab = 0$$

in each triangle

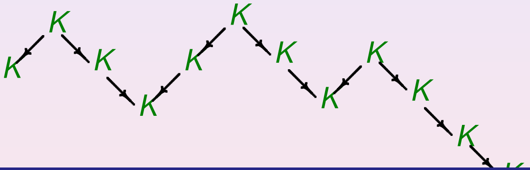
Bottom half has reversed orientation from cluster

String modules of finite length

Proposition

Any finitely generated module over $J(Q, W)$ is a sting module.

$mod\, J(Q, W)$: category of finitely generated $J(Q, W)$ modules.



Theorem

$$R/T \cong \underline{mod}\, J(Q, W) \cong f.l.mod\, J(Q, W)$$

$\underline{mod}\, J(Q, W) = mod\, J(Q, W)/Projectives$: the stable category.

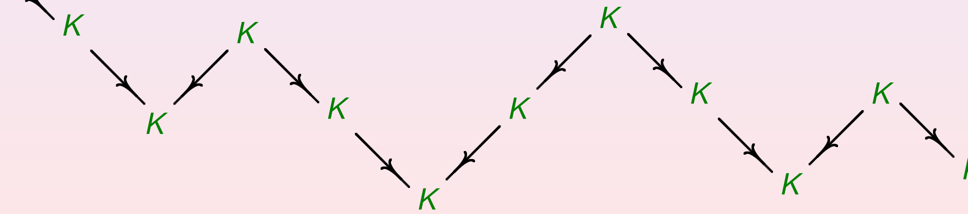
$f.l.mod\, J(Q, W)$: the category of module of finite length.

Infinite length string modules

Definition

$Rep_+ J(Q, W) =$ additive category generated by string modules with two infinite ends which are not of injective type.

$Rep_+ J(Q, W)$ contains all f.g. projective modules.



Last slide

Definition

$Rep_0 J(Q,W) =$ add {string modules with 0,1 or 2 infinite ends which are not of injective or projective type}

$Rep_0 J(Q,W)$ contains $f.l.mod\, J(Q,W)$

Theorem

$$C/T \cong \underline{Rep}_+ J(Q,W) \cong Rep_0 J(Q,W)$$