THE GENERALIZED GRASSMANN INVARIANT

\[ k_3(z[\pi]) \rightarrow H_0(\pi; z_2[\pi]) \]

by

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The purpose of this paper is two-fold. First, we give an elementary proof for the existence of an "exotic" element of \( K_3(Z) \). Second, we give a statement and partial proof of the final correct version of the Hatcher-Wagoner result on \( \pi_0 \Theta(M, \partial M) \) for \( \text{dim } M \geq 5 \).

The extra element of \( K_3(Z) \) is given explicitly as a 3-cycle in \( St(Z) \) which is derived in a natural way from a nonbounding 3-chain in \( W(\pm 1) \). This 3-cycle determines a homology class which is detected by an essentially geometric invariant \( \chi: K_3(Z) \to Z_2 \) which we call the Grassmann invariant. Since this invariant is zero on the image of \( H_3 W(\pm 1) \) we have a nontrivial element of \( K_3(Z) \).

The Grassmann invariant can be generalized to a homomorphism \( \chi: K_3(Z[\Pi]) \to K_1^+(Z[\Pi], Z_2[\Pi]) = H_0(\Pi; Z_2[\Pi]) \). The definition of \( \chi \) comes from very intuitive geometric considerations, but unfortunately the algebraic analogue is rather clumsy. Since the kernel of \( \chi \) contains the image of \( \Omega_3^{fr}(B\Pi) = \Pi_3^B(B\Pi \cup \text{pt.}) \) we can define a map on "Whitehead groups" \( \chi_{Wh}: Wh_3(\Pi) \to Wh_1^+(\Pi; Z_2) \) where \( Wh_3(\Pi) = K_3(Z[\Pi])/\Omega_3^{fr}(B\Pi) + K_3(Z) \) and \( Wh_1^+(\Pi; Z_2) = K_1^+(Z[\Pi], Z_2[\Pi])/K_1^+(Z, Z_2) \) where \( K_1^+(Z, Z_2) = Z_2 = \chi(K_3(Z)) \). The elements of \( Wh_1^+(\Pi; Z_2) \) in the image of \( \chi_{Wh} \) are those that die in pseudoisotopy. Thus if \( M \) is a compact manifold of dimension \( \geq 5 \) and \( \pi_1 M = \Pi, \pi_2 M = 0 \) we get an exact sequence

\[
Wh_3(\Pi) \to Wh_1^+(\Pi; Z_2) \to \pi_0 \Theta(M, \partial M) \to Wh_2(\Pi) \to 0.
\]

§1. The space of pictures for \( H_3 G \).

Suppose that \( G \) is a group with a presentation \( G = \langle \mathcal{F} | \mathcal{R} \rangle \). Then we
can construct a free $G$-resolution of $Z$ whose first three terms are

$$0 \leftarrow Z \leftarrow Z[G] \leftarrow \frac{Z[G]}{\mathcal{L}} \leftarrow \frac{Z[G]}{\mathcal{Y}}.$$ 

The groups and maps are defined by $Z[G]/\mathcal{L}$ is the free $G$-module generated by symbols $[x]$ where $x \in \mathcal{L}$.

$$\partial_1[x] = x - 1,$$

$$\partial_2[x_1, \ldots, x_n] = [x_1] + x_1[x_2] + \ldots + x_1x_2 \ldots x_{n-1}[x_n],$$

$$[x^{-1}] = x^{-1}[x] \text{ if } x \in \mathcal{L}.$$

$\varepsilon(g) = 1$ for all $g \in G \subseteq Z[G]$.

The exactness of the above sequence is well known and can be derived from the fact that it forms part of the augmented $G$-equivariant chain complex for $\widetilde{BG}$, the universal covering space for $BG$, where $BG$ is constructed in the obvious way with one 0-cell, a 1-cell for every element of $\mathcal{L}$, and a 2-cell for every element of $\mathcal{Y}$. One must of course add more cells of dimension $\geq 3$ but this can be done arbitrarily.

By considering long exact coefficient sequences we get

$$H_3G \cong H_2(G; \ker \varepsilon) \cong H_1(G; \ker \partial_1)$$

which fits into the exact sequence

$$0 \to H_3G \to H_2(G; \ker \partial_2) \to H_1(G; Z[G]/\mathcal{Y}) \to H_0(G; \ker \partial_1) \to 0.$$ 

Thus $H_3G$ is essentially contained in $\ker \partial_2$. We shall show that every element of $\ker \partial_2$ can be represented by a planar graph with certain labels on the edges and vertices. $\ker \partial_1$ can be identified with $\mathcal{R}/\mathcal{R'}$ where $\mathcal{R}$ is the kernel of the obvious map $\mathcal{Y} \to G$ where $\mathcal{Y}$ is the free group generated by $\mathcal{L}$.

We will need to assume that $\mathcal{Y}$ is a reduced set of relations for $G$. That is, $\mathcal{Y}$ and $\mathcal{Y}^1$ are disjoint in $\mathcal{Y}$. Since $1$ is the only element
of $F$ which is its own inverse, every set of relations contains a reduced set of relations.

**Definition 1.1:** Let $P(G)$ be the set of finite planar graphs together with the following additional data.

a) Every edge should be oriented and labeled with an element of $F$.

b) At every vertex we get an element of $F$ up to cyclic permutation by reading the labels of the incident edges in a counter-clockwise direction around the vertex, the label should be inverted if the corresponding edge is oriented outward. This word should be an element of $F$, or an inverse of an element of $F$ up to cyclic permutation of the letters.

c) In the case where different elements of $F$ are cyclic permutations of each other or when an element of $F$ is a nontrivial cyclic permutation of itself (e.g. $x_1 x_2 x_1 x_2$) a base point direction must be indicated at the vertex to indicate the starting position of the word.

d) Two graphs are equivalent if there is an orientation preserving self-homeomorphism of the plane which takes one graph to the other and preserves all the data above.

**Example 1.2:** Let $G = Z_2 = \langle x | x^2 \rangle$. The following graph with labels is an element of $P(G)$

![Diagram](attachment:image.png)
The asterisks indicate the base point directions at the two vertices. The relation at the top vertex is $x^2$ and at the bottom vertex it is $x^{-2}$.

$P(G)$ is a commutative monoid where addition is given by disjoint union and the empty graphs is the identity. By modding out an equivalence relation we shall make $P(G)$ into a G-module which is isomorphic to $\ker \delta_2$.

**Definition 1.3:** Let $\overline{P}(G)$ be the quotient of $P(G)$ by the following relations which we shall call deformations.

a) If a graph contains a circular edge with no vertices on it and nothing inside the circle, then this edge can be eliminated.

b) If two edges (or two portions of one edge) have the same label and opposite orientation they can be connected by a concordance if there is nothing between them.

\[
\begin{align*}
\cdots & \quad \cdots \\
\quad x & \quad \stackrel{\longrightarrow}{x} \\
\cdots & \quad \cdots
\end{align*}
\]

c) Two vertices can be cancelled if the associated relations are inverses of each other and if there is a path disjoint from the graph connecting the two base point directions.
$\overline{\mathcal{P}}(G)$ is an abelian group. The negative of a graph is given by its mirror image with the same labels but opposite orientations on the edges. $F$ acts on $\overline{\mathcal{P}}(G)$ on the left. The action of $x \in \mathfrak{X}$ on a graph is given by enclosing the graph by a circular edge oriented clockwise and labeled with $x$. The same procedure with counterclockwise orientation is the action of $x^{-1}$. The action of $xx^{-1}$ can easily be seen to be trivial by deformations (b), (a).

If $y \in \mathfrak{Y} \subset F$ and $P \in \overline{\mathcal{P}}(G)$ then the following deformation shows that $yP = P$.

$yP = \begin{array}{c}
\mathcal{P} \\
\circlearrowleft
\end{array}$

$\overset{(c)^{-1}}{\mathcal{P}} = \begin{array}{c}
\mathcal{P} \\
\circlearrowright
\end{array}$

$y = \begin{array}{c}
\mathcal{P} \\
\circlearrowright
\end{array}$

$\overset{y}{\mathcal{P}} = \begin{array}{c}
\mathcal{P} \\
\circlearrowright
\end{array}$

$y \circlearrowleft$
The second deformation is an isotopy which pushes $P$ through the gap between the two base point directions for $y, y^{-1}$. $\overline{P(G)}$ is thus a left $G$-module.

**Theorem 1.4:** If $\mathcal{Y}$ is a reduced set of relations for $G$ then $\overline{P(G)} \cong \ker \delta_2$ as $G$-modules.

**Proof:** We shall show that $\overline{P(G)}$ is the kernel of the natural map $Z[G] \mathcal{Y} \longrightarrow R/R'$. Then when we show that $R/R' \cong \ker \delta_1 = \text{im} \delta_2$ and that the two maps are consistent then we will know that $\overline{P(G)} \cong \ker \delta_2$.

**Definition 1.5:** Let $Q(G)$ represent the group generated by pairs $(f, y)$ where $f \in F$ and $y \in \mathcal{Y}$ modulo the relations

$$(*) \quad (f, y)(f', y')(f, y)^{-1} = (f y f^{-1} y', y').$$

Let $\varphi: Q(G) \longrightarrow R$ be the homomorphism given by $\varphi(f, y) = fyf^{-1}$.

**Lemma 1.6:** $\ker \varphi \cong \overline{P(G)}$ and it is contained in the center of $Q(G)$.

**Proof:** It is clear that $(*)$ centralizes the kernel of $\varphi$. Moreover since $R$ is centerless in most cases (i.e. unless $F$ has only one element) we have $\ker \varphi = ZQ(G)$. Since $R$ is free, $Q(G)$ is group isomorphic to $R \times \ker \varphi$. Since $\ker \varphi$ is always abelian, the commutator subgroups $Q(G)', R'$ are isomorphic and the isomorphism is induced by $\varphi$.

It is clear from $(*)$ that $Q(G)/Q(G)' \cong Z[G] \mathcal{Y}$ and that the induced map $Z[G] \mathcal{Y} \longrightarrow R/R'$ is the $G$-map which sends $y \in \mathcal{Y}$ to its $R'$ coset, $yR'$. 
An isomorphism \( \psi: P(G) \to \ker \varphi \) is given as follows. Let \( P \) be a graph representing an element of \( P(G) \). From every vertex of \( P \) draw a line from its base point direction to \( \infty \). These lines can be chosen to be disjoint from all other vertices and from each other. To each of these lines we will associate a pair \((f, y)\) where \( f \in P, \ y \in Y \cup Y^{-1} \). If we use the convention \((f, y^{-1}) = (f, y)^{-1}\) we get a generator of \( Q(G) \). By multiplying these together we get an element
\[
(f_1, y_1) \ldots (f_n, y_n) \in Q(G) \text{ if the lines are numbered in a clockwise direction near } \infty. \text{ This will be } \psi(P).
\]

The association of the pair \((f, y)\) is given as follows. \( y \) is just the relation given by the vertex. \( f \) is a product of elements of \( X \cup X^{-1} \) which when read from left to right give the labels for the edges which cross the line as we come from \( \infty \). The label is inverted if the edge is oriented from left to right across the line.

To show that \( \psi(P) \) is well defined we must show that it is independent of the choice of the lines and their ordering, and we must show that \( \psi(P) \) is invariant under deformations of \( P \). An isotopy of the lines does not change \( \psi(P) \) because the cyclic ordering of the lines is unchanged and the elements \( f_i \) cannot be changed without going through relations (at the vertices). If the \( i \)-th line passes through the \( i-1 \)-st vertex, i.e., if it is rechosen to go before the \( i-1 \)-st line, then \( f_i \) is changed to \( f_i-1 f_{i-1} f_{i-1}^{-1} f_i \) and \( \psi(P) \) is changed by the relation \((\ast)\). A sequence of such changes takes any choice to any other choice.

\( \psi(P) \) is invariant under cyclic permutation of the generators \( (f_1, y_1) \). This is easily seen to be true for any central element of \( Q(G) \). To see that \( \psi(P) \in \ker \varphi \subset ZQ(G) \) pull each vertex out to \( \infty \) along the attached
line. The edges will then read $f_1 y_1 f_1^{-1} \ldots f_n y_n f_n^{-1}$ clockwise near $\infty$.

Since the vertices are all gone this element is $1$ in $F$.

$\psi(P)$ is invariant under deformations of $P$. Deformations (a), (b) obviously don't matter. For deformation (c) choose the lines for the two canceling vertices adjacent to each other. Then they will contribute canceling elements $(f, y), (f, y^{-1})$ of $Q(G)$.

If $P$ is the disjoint union of two graphs $P_1, P_2$ we can put the graphs in separate half planes and choose the lines inside the respective half plane. Then we get $\psi(P) = \psi(P_1)\psi(P_2)$.

We shall now define $\psi^{-1} \colon \ker \varphi \longrightarrow \overline{F}(G)$. The graph for $\psi^{-1}((f_1', y_1') \ldots (f_n', y_n'))$ should have $n$ vertices. Take any $n$ distinct points in the plane and draw disjoint lines out to $\infty$. Using the lines as base point directions put in the necessary edges with labels and orientation at each end point. Using $f_i'$ put in the necessary edges with labels and orientation across the $i$-th line. Since $f_1 y_1 f_1^{-1} \ldots f_n y_n f_n^{-1} = 1$ in $F$, the loose ends of the partial edges can be connected together without intersections while respecting the labels and orientations of the edges. The last step involves a choice which is unique up to deformations (a), (b).

The proof that $\psi^{-1}$ is a well defined homomorphism is analogous to the proof for $\psi$. $\psi^{-1}(P)$ is concordant to $P$, i.e. they are equivalent modulo deformations (a), (b). If $z \in \ker \varphi$, $\psi^{-1}(z) = z$ is obvious from the definition of $\psi^{-1}$.

The $G$-equivariance of $\psi$ is left to the reader.

Lemma 1.7: $R/R' \cong \ker \varphi$ as $G$-modules.
Proof: Define a homomorphism $\delta : R \rightarrow \mathbb{Z}[G] \langle x \rangle$ with the same formula as $\delta_2$, i.e. if $r = x_1 \ldots x_n$, $\delta r = [x_1] + x_1 [x_2] + \ldots + x_1 \ldots x_{n-1} [x_n]$.

Then $\delta_1 r_2 = \delta r_1 + r_1 \delta r_2 = \delta r_1 + \delta r_2$. $\delta$ induces a homomorphism $\delta' : R/R' \rightarrow \mathbb{Z}[G] \langle x \rangle$ which is easily seen to be $G$-equivariant. One can also see easily that $\delta_2$ is given by the composition $\mathbb{Z}[G] \langle x \rangle \rightarrow R/R' \rightarrow \mathbb{Z}[G] \langle x \rangle$.

We shall give a "geometric" proof that $\delta'$ is an isomorphism. Ker $\delta_1$ can be interpreted as the group of $1$-cycles of the universal covering space $BG$ of $BG$ with the induced equivariant cell structure. Since $BG$ is the $1$-skeleton of $BG$, the $1$-skeleton of $\tilde{BG}$ is a classifying space for $R$.

Thus Ker $\delta_1 \cong H_1 R \cong R/R'$. That this isomorphism is given by $\delta'$ is straightforward.

This concludes the proof of (1.4).

Corollary 1.8: The following sequences are exact.

\begin{align*}
\text{a)} \quad & 0 \rightarrow H_3 G \rightarrow \mathbb{P}(G) \otimes \mathbb{Z} \rightarrow \mathbb{Z} \langle x \rangle \rightarrow R/R' \otimes \mathbb{Z} \rightarrow 0 \\
\text{b)} \quad & 0 \rightarrow H_2 G \rightarrow R/R' \otimes \mathbb{Z} \rightarrow \mathbb{Z} \langle x \rangle \rightarrow H_1 G \rightarrow 0
\end{align*}

where $\mathbb{Z} \langle x \rangle$ denotes the free abelian group generated by $x$.

Example 1.9: $G = \mathbb{Z}_2 = \langle x | x^2 \rangle$. The relevant exact sequence is

\[ 0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}[\mathbb{Z}_2] \leftarrow \mathbb{Z}[\mathbb{Z}_2] \leftarrow \mathbb{Z}[\mathbb{Z}_2] \leftarrow \mathbb{Z}[\mathbb{Z}_2] \leftarrow \mathbb{P}(\mathbb{Z}_2) \leftarrow 0 \]

Thus $\mathbb{P}(\mathbb{Z}_2)$ is the cyclic subgroup of $\mathbb{Z}[\mathbb{Z}_2]$ generated by $x - 1$. $\mathbb{Z}_2$ acts by negation. The generator of $\mathbb{P}(\mathbb{Z}_2)$ can be written as an element of $Q(\mathbb{Z}_2)$ by $(x, x^2)(1, x^2)^{-1}$ the corresponding graph for which is given
in (1.2). This is the only nontrivial element of $\mathbb{F}(\mathbb{Z}_2) \otimes \mathbb{Z}_2$ so it represents the generator of $H_3 \mathbb{Z}_2$.

§2. $K_3(\mathbb{Z}[\Pi])$ and $\mathbb{F}(\text{St}(\mathbb{Z}[\Pi]))$.

The inclusion $K_3(\mathbb{Z}[\Pi]) \cong H_3 \text{St}(\mathbb{Z}[\Pi]) \subset H_0(\text{St}(\mathbb{Z}[\Pi]); \mathbb{F}(\text{St}(\mathbb{Z}[\Pi])))$ has a natural section which is given by modding out "second order Steinberg relations." Thus every element of $\mathbb{F}(\text{St}(\mathbb{Z}[\Pi]))$ represents a well determined element of $K_3(\mathbb{Z}[\Pi])$. This can also be done for arbitrary rings but the procedure is more complicated because there are 25 percent more Steinberg relations and 80 percent more second order relations.

**Definition 2.1:** $\text{St}(\mathbb{Z}[\Pi]) = \langle \mathcal{Y} \rangle$ where

$\mathcal{X} = \{e_{ij}^u | i, j$ are distinct natural numbers and $u \in \Pi\}$

$\mathcal{Y} \cup \mathcal{Y}^{-1}$ consists of the following elements of $\mathbb{F}$.

1. $[e_{ij}^u, e_{kl}^v]$ where $i \neq j, j \neq k, u \neq v$ if $i = k$ and $j = k$.

2. $[e_{ij}^u, e_{jk}^v]e_{ik}^{-uv}$ if $i \neq k$. $e_{ik}^{-uv}$ represents $(e_{ik}^{uv})^{-1}$.

3. $e_{ik}^{uv}[e_{jk}^v, e_{ij}^u]$ if $i \neq k$.

$\mathcal{Y}$ will represent any maximal reduced subset of $\mathcal{Y} \cup \mathcal{Y}^{-1}$.

We shall not need an explicit choice of $\mathcal{Y}$ provided we make the following conventions. $\mathbb{Z}[\text{St}(\mathbb{Z}[\Pi])]\langle \mathcal{Y} \rangle$ will denote the $\text{St}(\mathbb{Z}[\Pi])$-module generated by $\mathcal{Y} \cup \mathcal{Y}^{-1}$ modulo the relation $[y] + [y^{-1}] = 0$. And similarly for $\mathbb{Z}\langle \mathcal{Y} \rangle$. 

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Since \( H_1 \text{St}(Z[\pi]) = 0 \) for \( i = 1, 2 \) (see [M]) (1.6) produces the exact sequence

\[ 0 \longrightarrow K_3(Z[\pi]) \longrightarrow \overline{\text{St}}(Z[\pi]) \otimes \mathbb{Z} \xrightarrow{h} \mathbb{Z}_{<} \xrightarrow{\delta} \mathbb{Z}_{<} \longrightarrow 0 \]

**Lemma 2.2:** The kernel of \( \delta : \mathbb{Z}_{<} \longrightarrow \mathbb{Z}_{<} \) is generated by the image under \( h \) of the following elements of \( \overline{\text{St}}(Z[\pi]) \).

a)

\[ e_{ij}^{u}, \quad e_{ik}^{v}, \quad e_{jk}^{w}, \quad e_{em}^{w} \quad \text{where} \quad e \neq j, k \]

m \neq i, j, k.

The exceptional case \( e = i, m = k, w = uv \) is explained in 2.3(0).

b)

\[ i, j, k, \ell \text{ distinct} \]
We are using the convention that a smooth sequence of edges without corners all have the same label and orientation.

**Proof:** Since \( \delta([a, b]) = 0 \), \( \delta(c[a, b]) = [c] \), and \( \delta([a, b]c^{-1}) = -[c] \), the kernel of \( \delta : \mathbb{Z} \langle \mathcal{P} \rangle \twoheadrightarrow \mathbb{Z} \langle \mathcal{X} \rangle \) is generated by the relations 2.1(1) and pairs of relations one from 2.1(2) and one from 2.1(3) with cancelling boundaries. If we apply \( h \) to 2.2(a) we get \([e^w_{e_i}, e^{uv}_{e_{ik}}]\) which is a generator of \( \ker \delta \) of the first kind and if we apply \( h \) to 2.2(b) we get \([e^{u}_{i}, e^{vw}_{e_{ij}}, e^{uv}_{e_{ik}}]\) + more generators of the first kind. This is a generator of the second kind.

Let \( H_0 \) be the submodule of \( \overline{P}(\text{St}(\mathbb{Z}[\Pi])) \) generated by elements of the form 2.2(a), (b). It is clear from (2.2) that the composition \( \overline{K}_3(\mathbb{Z}[\Pi]) \xrightarrow{\text{St}} \overline{P} \otimes \mathbb{Z} \twoheadrightarrow \overline{P}/H_0 \otimes \mathbb{Z} \) is surjective. One can show that it is in fact an isomorphism.

**Definition 2.3:** Let \( H \) be the submodule of \( \overline{P}(\text{St}(\mathbb{Z}[\Pi])) \) generated by the following second order Steinberg relations.

\[
\begin{align*}
\delta^u_{ij} & \\
\end{align*}
\]

This is not an element of \( \overline{P} \) since the relation \([e^u_{i}, e^u_{ij}]\) is not allowed.

This should on the other hand be thought of as a convention which says that whenever a general formula includes a relation of the form \([x, x]\) in one of its special cases this vertex should be deleted and the graph modified in the following way.
(1) $e_{ij}^u e_{kl}^v e_{mn}^w$ \quad \text{j} \neq k$, etc.

(2) Same as 2.2(a)

(3) $e_{ij}^u e_{ik}^v e_{jk}^x$ \quad i \neq k, \ell

(4) $e_{ij}^u e_{ik}^v e_{jk}^x$ \quad k \neq i, \ell
(5) Same as 2.2(b)

The following theorem and its partial proof can be ignored for the purpose of the remainder of this paper.

**Theorem 2.4:** The composition

$$K_3(\mathbb{Z}[[\pi]]) \rightarrow \overline{p} \otimes \mathbb{Z} \rightarrow \overline{p}/\overline{H} \otimes \mathbb{Z}$$

is an isomorphism.

**Proof:** Since $H \supset H_0$, we know that the composition above is surjective. Thus it is sufficient to show that the image of $H \otimes \mathbb{Z}$ in $\overline{p} \otimes \mathbb{Z}$ is disjoint from the image of $K_3(\mathbb{Z}[[\pi]])$. This can be reworded as follows. We must show that every element of $H$ which goes to zero in $\overline{p} \otimes \mathbb{Z}$ is already zero in $\overline{p} \otimes \mathbb{Z}$.

**Lemma 2.5:** An element of $\overline{p}$ of the form 2.3(1) is zero in $\overline{p} \otimes \mathbb{Z}$.

**Proof:** Let $p$ be an element of the form 2.3(2) = 2.2(a). Let $X$ be a generator of $St(\mathbb{Z}[[\pi]])$ which commutes with the four generators involved in $p$. We shall perform a deformation on $xp - p$. 
The circular edge in (4) can be removed because $c^{-1}g - g$ is zero in $\mathbb{P} \otimes \mathbb{Z}$. When (4) is then attached at the top of (1) it cancels the small $x$ circle at the top. After multiplying (2) by $b^{-1}$ it can be attached to (1) on the right to cancel the right hand little $x$ circle. Similarly $a^{-1}(3)$ will cancel the left hand little $x$ circle and we are left with:

Eliminating the two circular edges labeled $a$, $b$ produces a typical element of the form 2.3(1).

The remainder of the proof is similar. The details can be found in [I].

83. The Generalized Grassmann Invariant.

We shall define a natural homomorphism
\( x: K_3(\mathbb{Z}[\pi]) \longrightarrow K_1(\mathbb{Z}[\pi]; \mathbb{Z}_2[\pi]) \cong H_0(\pi; \mathbb{Z}_2[\pi]) \)

where \( \pi \) acts on \( \mathbb{Z}_2[\pi] \) by conjugation.

**Definition 3.1:** The intersection pairing will be the symmetric biadditive map \( \mathbb{Z}[\pi] \times \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \) given by the formula

\[
\langle \sum_{1} u_1, \sum_{1} u_1 \rangle = \sum_{1} u_1 \cdot u_1
\]

We shall also use the same notation for the induced pairing 
\( \mathbb{Z}_2[\pi] \times \mathbb{Z}_2[\pi] \longrightarrow \mathbb{Z}_2[\pi] \) which can be interpreted as set intersection if we think of \( \mathbb{Z}_2[\pi] \) as the set of finite subsets of \( \pi \).

This pairing clearly satisfies the following condition. If \( u, v \in \pi \) and \( x, y \in \mathbb{Z}[\pi] \) or \( \mathbb{Z}_2[\pi] \) then \( \langle uxv, uyv \rangle = u\langle x, y \rangle^v \).

Let \( \chi_\pi: Q(\text{St}(\mathbb{Z}[\pi])) \longrightarrow H_0(\pi; \mathbb{Z}_2[\pi]) \) be defined by the following formula

\[
\chi_\pi(f, [e^u_{i,j}, e^v_{i,k}]) = \sum_{p} r_{pi} \langle u_{jp}, v_{kp} \rangle
\]

where \( (r_{pq}) \) is a matrix representing the image of \( f \) in \( GL(\mathbb{Z}_2[\pi]) \) and \( (s_{pq}) \) is its inverse.

\( \chi_\pi(f, y) = 0 \) if \( y \) is not of the above form.

Since the range is abelian \( \chi_\pi \) induces an additive homomorphism

\[ \chi_c: \mathbb{Z}[\text{St}(\mathbb{Z}[\pi])] \langle y \rangle \longrightarrow H_0(\pi; \mathbb{Z}_2[\pi]). \]

By restricting this map to \( \ker \beta_{2} \) we get an additive homomorphism

\[ \chi: \overline{F}(\text{St}(\mathbb{Z}[\pi])) \longrightarrow H_0(\pi; \mathbb{Z}_2[\pi]). \]
We shall show that \( \chi_p \) is a homomorphism of \( \text{St}(\mathbb{Z}[H]) \) modules and that it is zero when restricted to \( H_o \).

**Theorem 3.2:** \( \chi_p(H_o) = 0 \).

**Proof:** The following computations can also be carried out for all the second order Steinberg relations to show \( \chi_p(H) = 0 \).

a) Suppose that \( fP \) is an additive generator of \( H_o \) where \( f \in F \) and \( P \) is a graph of the form 2.2(a). If \( \ell \neq i \) then

\[
\chi_p(fP) = 0 \quad \text{because no relevant relations exist. If} \quad \ell = i \quad \text{then there are three relevant relations except in the exceptional case when} \quad m = k \quad \text{and} \quad w = uv. \quad \text{This exceptional case is taken care of by c) below.}
\]

At the 3 relevant relations in the case \( \ell = i \) we have:

1) \( \chi_q(f \beta^{-u} e_{ij}^{-u}, [e_{im}^u, e_{ij}^u]) = \sum_p r_p <w(s_{jp} + vs_{kp}), ws_{mp}> \)

2) \( \chi_q(f \beta^{-v} e_{ik}^{-u}, [e_{ij}^u, e_{im}^u]) = \sum_p r_p <u(s_{jp} + vs_{kp}), ws_{mp}> \)

3) \( \chi_q(f \beta^{-v} e_{ik}^{-v}, [e_{ij}^w, e_{ik}^w]) = \sum_p r_p <w(s_{jp} + vs_{kp}), ws_{mp}, uv(s_{kp})> \)

where \((s_{**})^{-1} = (r_{**})\) is the image of \( f \) in \( \text{GL}(\mathbb{Z}_2[\Pi]) \).

It can easily be seen that \((2) = (1) + (3)\) so the sum of all three is zero.

b) \( \chi_p(fP) = 0 \) if \( P \) is a graph of the form 2.2(b). There are no relevant Steinberg relations.

c) Let us pretend for a moment that we are allowing \([x, x]\) as a Steinberg relation. Then we have \( \chi_q(f, [e_{ij}^u, e_{ij}^u]) = \sum_p r_p <us_{jp}, us_{jp}> = \sum_p r_p us_{jp} = \sum_p us_{jp} r_p = 0 \) since \( i \neq j \).

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This means that the general computation of a) extends to the exceptional case.

Theorem 3.3: \( \chi: \bar{G} \rightarrow H_0(\Pi; \mathbb{Z}_2[\Pi]) \) is a homomorphism of \( \text{St}(\mathbb{Z}[[\Pi]]) \)-modules where the action on \( H_0(\Pi; \mathbb{Z}_2[\Pi]) \) is trivial.

Proof. Let \( x = e_h^w \) be an arbitrary generator of \( \text{St}(\mathbb{Z}[[\Pi]]) \). We shall show that \( \chi_c(x - 1) \) is an additive coboundary, i.e. it factors through \( \mathbb{Z}[\text{St}] < \mathcal{E} > \), and thus \( \chi_c(x - 1) = 0 \).

\[
\chi_c(x - 1)(f[[e_{ij}^u, e_{ik}^v]]) = \chi_c(f[[e_{ij}^u, e_{ik}^v]]) + \chi_c(e_h^w f[[e_{ij}^u, e_{ik}^v]]) = \sum \frac{r_i}{p} \langle u_i^j, v_i^k > + \sum \frac{r_i'}{p} \langle u_i'^j, v_i'^k > \]

where \( r_i' = r_i \) if \( p \neq h \)

\[ r_i' = r_i + w_i^l \]

\[ a_p' = a_p \]

Thus the terms in the above sum cancel except when \( p = h \) or \( l \) where we get (\( p = h \)): \( wr_i^j < u_i^j, v_i^k > \)

(\( p = l \)): \( r_i^l < u_i^j, v_i^k > + vs_i^j w_i^k + vs_i^k w_i^j \)

since \( r_i^j u_i^j, v_i^k > = r_i^l u_i^j, v_i^k > = wr_i^j v_i^k > \), we have left only two of the four cross terms of \( p = l \) which are

**Formula 3.4:** \( \chi_c(e_h^w - 1)(f[[e_{ij}^u, e_{ik}^v]]) = r_i^l < u_i^j, v_i^k > + r_i^l < u_i^j, v_i^k > \)
Let \( \Psi : \mathbb{Z} [\text{St}(\mathbb{Z}[\eta])] \xrightarrow{\mathcal{E}} H_0(\eta; \mathbb{Z}_2[\eta]) \) be defined by the following formula.

\[
\Psi(f[e^u_{i,j}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + us_{j} w >
\]

if \( u \in \mathcal{F}. \) From the convention \([x^{-1}] = -x^{-1}[x]\) we get the formula

\[
\Psi(f[e^{-u}_{i,j}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w >.
\]

We shall now show that \( \Psi_2 = \chi_c(x - 1) \). Note that \( \chi_c \) is equivariant but the other homomorphisms are not.

a) \( \Psi_2 f[[e^u_{i,j}, e^v_{k,m}]] \) is a sum of four terms.

1) \( \Psi(f[e^u_{i,j}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + us_{j} w > \)

2) \( \Psi(f[e^v_{i,k}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + us_{j} w + vs_{kh} w > \)

3) \( \Psi(f[e^u_{i,j} e^v_{i,k}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + vs_{kh} w + us_{j} w > \)

4) \( \Psi(f[e^v_{i,k} e^u_{i,j}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + vs_{kh} w > \)

All the terms cancel except two which are the same as the terms given in (3,4).

b) \( \Psi_2 f[[e^u_{i,j}, e^v_{k,m}]] = 0 \) if \( i \neq k, m \) and \( j \neq k \). This follows from the general formula \( \Psi(f[e^v_{k,m}]) = \Psi(f[e^u_{i,j}]) \) if \( i \neq k, m \) and \( j \neq k \).

c) \( \Psi_2 f[[e^u_{i,j}, e^v_{j,k} e^{-u}_{i,k}]] \) is the sum of the following five terms.

1) \( \Psi(f[e^u_{i,j}]) = r_{\mathcal{E} i} < us_{j', s_{ih}} w + us_{j} w > \)

2) \( \Psi(f[e^v_{j,k}]) = (r_{\mathcal{E} j} + r_{\mathcal{E} i} u) < vs_{k', s_{jh}} w + vs_{kh} w > \)

3) \( \Psi(f[e^u_{i,j} e^v_{j,k}]) = \Psi(f[e^v_{j,k} e^u_{i,j}]) = r_{\mathcal{E} i} < us_{j'} + vs_{k', s_{ih}} w + us_{j} w > \)

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4) \( \psi(f_{ik} e_{jk}^{UV} e_{jk}^{-1}) = \psi(f_{ik}^{UV} e_{jk}^{-1}) = r_{ik}^{<vs_{k}, s_{ih} w + vs_{k} w>}
\)

5) \( \psi(f_{ik}^{UV} e_{ik}^{-1}) = r_{ik}^{<uv_{s_k}, s_{ih} w + uv_{s_k} w>}
\)

One can easily see that all the terms cancel.

**Corollary 3.5:** \( \chi_\Pi \) is a 3-cocycle representing an element of \( H^3(St(Z[[\Pi]]) ; H_0(\Pi ; \mathbb{Z}_2[[\Pi]]) \).

This can be interpreted as a Postnikov invariant for Waldhausen's space \( GL(Z[[\Omega_3]]) \)
\( [W] \).

\[\text{§4. Naturality of } \chi.\]

In this section we shall prove that \( \chi \) is a natural transformation of functors on the category of groups and homomorphisms.

**Lemma 4.1:** \( \chi \) is natural for injective homomorphisms.

**Proof:** The intersection pairing is natural for injective homomorphisms.

Every homomorphism \( A \rightarrow B \) is the composition of a monomorphism \( A \rightarrow A \times B \) and a projection \( A \times B \rightarrow B \). Thus we will restrict our attention to the latter. As in the proof of (3,3) we shall show that the difference between the two maps

\[ Z[St(Z[[A \times B]])] \xrightarrow{\chi_\Pi} H_0(A \times B; \mathbb{Z}_2[A \times B]) \rightarrow H_0(B; \mathbb{Z}_2[B]) \]

and
\[ \mathbb{Z}[\text{St}(\mathbb{Z}[A \times B])]<\Phi> \rightarrow \mathbb{Z}[\text{St}(\mathbb{Z}[B])]<\Phi> \rightarrow \mathbf{H}_0(B; \mathbb{Z}_2[B]) \]

is an integral coboundary and thus zero on \( \overline{\Phi}(\text{St}(\mathbb{Z}[A \times B])) \).

This difference map will be denoted by \( \Delta \).

If \( x \in \mathbb{Z}_2[A \times B] \) then the image of \( x \) in \( \mathbb{Z}_2[B] \) will be denoted by \( \overline{x} \). If we define a symmetric biadditive pairing

\[ d: \mathbb{Z}_2[A \times B] \times \mathbb{Z}_2[A \times B] \rightarrow \mathbb{Z}_2[B] \text{ by } d(x, y) = <\overline{x}, \overline{y}> + <\overline{-x}, \overline{y}> \]

then we have

Formula 4.2: \( \Delta(f[[\varepsilon_{ij}, \varepsilon_{ik}^v]]) = \sum \overline{p} \cdot d(\varepsilon_{ij}^p, \varepsilon_{ik}^p) \)

We shall consider \( \mathbb{Z}_2[A] \) as the set of finite subsets of \( A \). If \( x \in \mathbb{Z}_2[A] \), \( |x| \) will represent the number of elements of \( x \). Let \( (x) \in \mathbb{Z}_2 \) be defined by

\[ f_h(x) = \begin{cases} 0 & \text{if } |x| = 0, 1 \ (4) \\ 1 & \text{if } |x| = 2, 3 \ (4) \end{cases} \]

Then we have the following formula

1) \( f_h(x + y) = f_h(x) + f_h(y) + |x| \cdot |y| + |x \cap y| \)

where \( x \cap y \) can also be written as \( <x, y> \). If \( x \in \mathbb{Z}_2[A \times B] \), we can define \( |x| = \sum_{b \in B} |x|_b \) where \( |x|_b = |xb^{-1} \cap A| \). Define \( f_h(x) \in \mathbb{Z}_2[B] \) by \( f_h(x) = \sum_{b \in B} f_h(x)_b \) where \( f_h(x)_b = f_h(xb^{-1} \cap A) \).

Then formula (1) generalizes to

2) \( f_h(x + y) = f_h(x) + f_h(y) + d(x, y) \).

Now let \( A \) be well ordered. If \( x \in \mathbb{Z}_2[A] \) let
$0^+(x) = (x_1, x_2, \ldots, x_n)$ be the elements of $x$ in increasing order. Let $0^-(x)$ be the same elements in decreasing order.

If $x, y$ are disjoint elements of $\mathbb{Z}_2[A]$, i.e. if $\langle x, y \rangle = 0$, then let $n_1(x, y) = 0, 1 \in \mathbb{Z}_2$ depending on whether $0^-(x) 0^+(x)$ and $0^+(x + y)$ differ by an even or odd permutation.

The function $\mathcal{R}$ originated in the following equation.

3) $n_1(y, x) = n_1(x, y) + \mathcal{R}(x + y)$.

If $x, y$ are disjoint elements of $\mathbb{Z}_2[A \times B]$, $n_1$ can be generated to $n_2(x, y) = \sum_{b \in B} n_1(xb^{-1} \cap A, yb^{-1} \cap A)b$. Then formula (3) becomes

4) $n_2(y, x) = n_2(x, y) + \mathcal{R}(x + y)$.

If $x, y$ are arbitrary elements of $\mathbb{Z}_2[A \times B]$ then they determine three mutually disjoint elements

$$x \setminus y = x + \langle x, y \rangle = \langle x, x + y \rangle$$

$$x \setminus y = y + \langle x, y \rangle = \langle y, x + y \rangle$$

$$x \cap y = \langle x, y \rangle$$

and we can define the following generalization of

$$n_2 \cdot n_3(x, y) = n_2(x \setminus y, y \setminus x) + n_2(x \cap y, x \setminus y) + n_2(x \cap y, y \setminus x).$$

Since $x + y = (x \setminus y) + (y \setminus x)$ formula (4) gives

5) $n_3(y, x) = n_3(x, y) + \mathcal{R}(x + y)$.

Let $\tilde{a}(x, y) = n_3(x, y) + \mathcal{R}(x)$, then 2) and 5) give

6) $\tilde{a}(x, y) = \tilde{a}(x, y) + \tilde{a}(y, x)$. 

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Lemma 4.3:
\[ \tilde{d}: \mathbb{Z}_2[A \times B] \times \mathbb{Z}_2[A \times B] \to \mathbb{Z}_2[B] \]

is biadditive.

Proof: Let \( \tilde{d}_1(x, y) = n_1(x, y) + n_1(y, x) \). Then \( \tilde{d}_1(x, y) \) is the parity of the permutation taking \( 0^+(x) \) \( 0^+(y) \) to \( 0^+(x + y) \). For each element of \( y \) count the number of elements of \( x \) that are larger and add these up for all the elements of \( y \) and one gets \( \tilde{d}_1(x, y) \). From this description one easily sees that \( \tilde{d}_1(x, y) \) is biadditive where defined. By the analogous argument at each \( b \in B \) one sees that \( \tilde{d}_2 \) is biadditive where defined. Using the biadditivity of \( \tilde{d}_2 \) and formula (2), the biadditivity of \( \tilde{d}_3 \) is a straightforward computation. Note that because of (6) one need only show additivity in one variable.

In order to force equivariance on \( \tilde{d}_3 \) we make the following definition.

\[ \tilde{d}_4(x, y, z) = \Sigma_{i} \tilde{d}_3(x_i, y, z_i) \text{ if } x = \Sigma_i x_i \text{ where } x_i \in A \times B. \]

Then we have the equivariance condition

7) \( \tilde{d}_4(xu, y, z) = \tilde{d}_4(x, uy, uz) \) if \( u \in A \times B. \)

Equation (6) now becomes

8) \( \tilde{d}_4(y, z) = \tilde{d}_4(x, y, z) + \tilde{d}_4(x, z, y). \)

Note that \( \tilde{d}_4 \) is additive in each variable.

Let \( \psi: \mathbb{Z}[\text{St}(\mathbb{Z}[A \times B])]<x> \to H_0(B; \mathbb{Z}_2[B]) \) be defined by

\[ \psi(f[e_{ij}^u]) = \Sigma_{p} \tilde{d}_4(r_{pi}^u, s_{jp}^u, s_{jp}^u) \text{ if } u \in A \times B. \]

We are using the notation \( (r_{pq}^{-1})^{-1} = \text{image of } f \text{ in } GL(\mathbb{Z}_2[A \times B]). \) The convention \( [x^{-1}] = -x^{-1}[x] \) necessitates the equation

\[ \psi(f[e_{ij}^{-u}]) = \Sigma_{p} \tilde{d}_4(r_{pi}^{-u}, s_{jp}^{-u}, s_{jp}^{-u}). \]

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Theorem 4.4: $\psi_2 = \Delta$.

Proof: We shall verify the equation on each additive generator of $\mathbb{Z}[\text{St}(\mathbb{Z} [A \times B])]$.

1) $\psi(\text{fe}_{ij}^u) = \sum_{p} \tilde{a}_4 (r_{pi}, u s_{jp}, s_{ip} + u s_{jp})$
2) $\psi(\text{fe}_{ij}^u \text{e}_{ik}^v) = \sum_{p} \tilde{a}_4 (r_{pi}, v s_{kp}, s_{ip} + u s_{jp} + v s_{kp})$
3) $\psi(\text{fe}_{ij}^u \text{e}_{ik}^v \text{e}_{ij}^{-u}) = \sum_{p} \tilde{a}_4 (r_{pi}, u s_{jp}, s_{ip} + u s_{jp} + v s_{kp})$
4) $\psi(\text{fe}_{ik}^u \text{e}_{ik}^v) = \sum_{p} \tilde{a}_4 (r_{pi}, v s_{kp}, s_{ip} + v s_{kp})$

The sum is easily seen to be equal to

$$\sum_{p} \tilde{r}_{pi} d(u s_{jp}, v s_{kp}) = \Delta(\text{fe}_{ij}^u, \text{e}_{ik}^v).$$

b) $\psi_2(\text{fe}_{ij}^u, \text{e}_{ik}^v) = 0$ if $i \neq k$, $l$ and $j \neq k$ this is the result of the general formula

$$\psi(\text{fe}_{ij}^u, \text{e}_{il}^v) = \psi(\text{fe}_{il}^v)$$

if $i \neq k$, $l$ and $j \neq k$.

c) $\psi_2(\text{fe}_{ij}^u, \text{e}_{ik}^v, \text{e}_{ik}^{-u} \text{e}_{ij}^{-v})$ is the sum of five terms

1) $\psi(\text{fe}_{ij}^u) = \sum_{p} \tilde{a}_4 (r_{pi}, u s_{jp}, s_{ip} + u s_{jp})$
2) $\psi(\text{fe}_{ij}^u \text{e}_{jk}^v) = \sum_{p} \tilde{a}_4 (r_{pj} + r_{pi} u, v s_{kp}, s_{jp} + s_{kp})$
3) $\psi(\text{fe}_{ij}^u \text{e}_{jk}^v \text{e}_{ij}^{-u}) = \sum_{p} \tilde{a}_4 (r_{pi}, u s_{jp} + v s_{kp}, s_{ip} + u s_{jp})$
4) $\psi(\text{fe}_{ik}^u \text{e}_{jk}^v \text{e}_{ik}^{-v}) = \sum_{p} \tilde{a}_4 (r_{pj}, v s_{kp}, s_{jp} + v s_{kp})$
5) $\psi(\text{fe}_{ik}^u \text{e}_{ik}^v \text{e}_{ik}^{-u} \text{e}_{ij}^{-v}) = \sum_{p} \tilde{a}_4 (r_{pi}, v s_{kp}, s_{ip} + v s_{kp})$

Using (7) and the triadditivity of $\tilde{a}_4$ this sum is easily seen to be zero.
§5. $\chi(\Pi^S_3(B\Pi \cup pt)) = 0.$

We will assume the reader is familiar with [M, §5].

Let $M(\Pi)$ denote the subgroup of $GL(\mathbb{Z}[\Pi])$ of monomial matrices with entries in $\Pi$.

**Proposition 5.1:** The commutative subgroup of $M(\Pi)$ is perfect and consists of all even monomials with abelianized determinant equal to 0 in $\Xi \Pi/\Pi'$.

Thus $M(\Pi)'$ admits a universal central extension $T(\Pi) \longrightarrow M(\Pi)'$ and there exists a unique homomorphism $T(\Pi) \longrightarrow St(\mathbb{Z}[\Pi])$ over the inclusion $M(\Pi)' \leq E(\mathbb{Z}[\Pi]) = GL(\mathbb{Z}[\Pi])'$.

By an argument analogous to the one in [G] one sees that $H_3 T(\Pi) \cong H_3 EM(\Pi)^+$ which is isomorphic to $\Pi^S_3(B\Pi \cup pt)$ by the generalized Kaln-Priddy theorem. We shall show that the image of $H_3 T(\Pi)$ in $H_3 St(\mathbb{Z}[\Pi]) = K_3(\mathbb{Z}[\Pi])$ is contained in the kernel of $\chi$.

**Definition 5.2:** Let $W(\Xi \Pi)$ be the group generated by symbols $w_{ij}(u)$ where $i, j$ are distinct natural numbers and $u \in \Pi$ modulo the reduced set of relations

1) $[w_{ij}(u), w_{kl}(v)]$ if $i, j, k, l$ are distinct and $i < k$.
2) $w_{ij}(u) w_{ik}(v) w_{ij}(u)^{-1} w_{jk}(u^{-1}v) \neq k$.
3) $w_{ij}(u) w_{ki}(v) w_{ij}(u)^{-1} w_{kj}(vu) \neq k$.
4) $w_{ij}(u) w_{jk}(v) w_{ij}(u)^{-1} w_{ik}(uv)^{-1} \neq k$.
5) $w_{ij}(u) w_{ki}(v) w_{ij}(u)^{-1} w_{ki}(vu^{-1})^{-1} \neq k$.

Let $\phi: W(\Xi \Pi) \longrightarrow M(\Xi \Pi)$ be the homomorphism given as follows.
φ(w_{ij}(u)) is the monomial matrix given by taking the identity matrix, multiplying the i-th column by u and the j-th column by -u^{-1} and transposing the two columns.

Lemma 5.3: The kernel of φ is contained in the center of \( W(\pm \Pi) \), i.e. \( W(\pm \Pi) \) is a central extension of \( \text{im} \phi \).

Proof: Let \( x \in \ker \phi \). We shall show that \( x \) commutes with \( w_{ki}(u) \).
Let \( j \) be a natural number which does not appear as an index in the expansion of \( x \) as a product of generators. Then by relation 5 we have \( w_{ki}(u) = w_{ij}(l) w_{kj}(u) w_{ij}(l)^{-1} \). Thus it is sufficient to show that \( x w_{kj}(u)x^{-1} = w_{kj}(u) \). However it is clear from the relations that \( x w_{kj}(u)x^{-1} = w_{kj}(v)^{\pm 1} \). Since \( \varphi(w_{kj}(u)) = \varphi(w_{kj}(v)^{\pm 1}) \) we must have \( w_{kj}(u) = w_{kj}(v)^{\pm 1} \).

Lemma 5.4: The image of \( \phi \) consists of all monomials in \( M(\pm \Pi) \) with abelianized determinant \( +1 \in \pm \Pi/\Pi' \).

Theorem 5.5: There exists a unique homomorphism \( T(\Pi) \longrightarrow W(\pm \Pi) \) covering the inclusion \( M(\Pi)' \subset M(\pm \Pi) \).

Proof: This follows from the universality of \( T(\Pi) \) and the above two lemmas.
Lemma 5.6: There exists a homomorphism $h: \mathcal{W}(\pm \Pi) \longrightarrow \text{St}(\mathbb{Z}[\Pi])^*$ covering the inclusion $M(\pm \Pi) \subset \text{GL}(\mathbb{Z}[\Pi])$.

Proof: Let $h(w_{ij}(u)) = e_{ij}^u e_{ji}^{-u^{-1}} e_{ij}^u$. For a proof that this is a homomorphism see [M, p. 72], or see §6.

Theorem 5.7: The image of $H_3 T(\Pi)$ in $H_3 \text{St}(\mathbb{Z}[\Pi]) = K_3(\mathbb{Z}[\Pi])$ is contained in the image of $h_x: H_3 \mathcal{W}(\pm \Pi) \longrightarrow H_3 \text{St}(\mathbb{Z}[\Pi])$.

Proof: By universality the map $T(\Pi) \longrightarrow \text{St}(\mathbb{Z}[\Pi])$ is equal to the composition $T(\Pi) \longrightarrow W(\Pi) \longrightarrow \text{St}(\mathbb{Z}[\Pi])$.

We shall consider $\chi$ as a cohomology class and show that $h^*(\chi) = 0$. To compute $h^*(\chi)$ take any equivariant chain map:

$$
\begin{array}{cccccccc}
0 & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z}[W] & \leftarrow & \mathbb{Z}[W]<\chi> & \leftarrow & \mathbb{Z}[W]<\Phi> & \leftarrow & P(W) & \leftarrow & 0 \\
& & & \downarrow l & & & \downarrow h_0 & & & \downarrow h_1 & & & \downarrow h_2 & & & \downarrow h_3 & \\
0 & \leftarrow & Z & \leftarrow & Z[St] & \leftarrow & Z[St]<\chi_{st}> & \leftarrow & Z[St]<\Phi_{st}> & \leftarrow & P(St) & \leftarrow & 0 \\
\end{array}
$$

The equivariant 3-cocycle $\chi_{\bar{P}(St)}$ which represents $\chi$ was defined as the \text{*}

\text{It is proved in [I] that this map is injective for finitely presented $\Pi$ thus justifying the notation.}

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coboundary of an integral (i.e. nonequivariant) 2-cochain

\[ \chi_c : \mathbb{Z} [St] \to \mathbb{Z}_2 [\Pi] \to H_c(\Pi; \mathbb{Z}_2 [\Pi]). \]

We shall show that \( \chi_c h_2 \) is \( W(\pm \Pi) \)-equivariant. This implies

\[ \chi_{\frac{\gamma}{\mathcal{P}(St)}} h_3 = \chi_c \delta_3 h_3 = \chi_c h_2 \delta_3 \]

is an equivariant coboundary and thus represents the trivial cohomology class.

Lemma 5.8: Let \( y \in \mathcal{Y}_S \), \( f \in St(\mathbb{Z}[\Pi]) \), and \( w \in W(\pm \Pi) \). Then

\[ \chi_c(h(w)f[y]) = \chi_c(f[y]). \]

Proof: These are both zero by definition unless \( y \) is a Steinberg relation of the form \( y = [e^u_{ij}, e^v_{ik}] \). In this case we have

\[ \chi_c(f[y]) = \frac{\Sigma}{p} r_{pi} <us_{jp}, vs_{kp}> \]

\[ \chi_c(h(w)f[y]) = \frac{\Sigma}{p} r'_{pi} <us'_{jp}, vs'_{kp}> \]

where \( (r^-1_{pq}) = (s_{qp})^{-1} \) is the image of \( f \) in \( GL(\mathbb{Z}_2 [\Pi]) \) and

\( (r'_{pq}) = (s'_{qp})^{-1} \) is the image of \( h(w)f \) in \( GL(\mathbb{Z}_2 [\Pi]) \). Every monomial matrix can be written uniquely as the product of a permutation matrix and a diagonal matrix. Thus \( \varphi(w) = PD \). Let the entries of the diagonal matrix \( D \) be written as \( d_p \) and let \( P \) be the permutation matrix gotten by permuting the rows of the identity matrix by \( \sigma^{-1} \). Then

\[ (r'_{pq}) = PD(r_{pq}) \quad (s'_{pq}) = D^{-1} p^{-1} \quad s'_{pq} = s_{qp} d^{-1} \sigma(p) \]
Thus we have $\chi_\mu(h(w)r[y]) = \sum \frac{r_\sigma(p)}{\sigma(p)} \langle \mu, j_{\sigma(p)} \frac{d_{\sigma(p)}}{\sigma(p)} \rangle \langle \nu_{\sigma(p)} \frac{d_{\sigma(p)}}{\sigma(p)} \rangle$

$= \sum \frac{r_\sigma(p)}{\sigma(p)} \langle \mu, j_{\sigma(p)} \rangle \langle \nu_{\sigma(p)} \rangle$ because $d_{\sigma(p)} \in \pm 1$. It is clear that the last expression is equal to $\chi_c(r[y])$.

**Theorem 5.9:** $h^*(\chi) = 0$ and thus the kernel of $\chi$ contains the image of $H_3(W(\pm 1))$.

§6. $\chi: K_3(\mathbb{Z}) \rightarrow \mathbb{Z}_2$ is surjective.

Lemma 2.2 implies that the image of $\chi: K_3(\mathbb{Z}) \rightarrow \mathbb{Z}_2$ is the same as the image of $\chi_\mu: \bar{F}(\text{St}(\mathbb{Z})) \rightarrow \mathbb{Z}_2$. We shall show that $\chi_\mu: \bar{F}(W(\pm 1)) \rightarrow \mathbb{Z}_2$ is surjective for an appropriate choice of the chain map $h_\chi$.

Let $h_0: \mathbb{Z}[W(\pm 1)] \rightarrow \mathbb{Z}[\text{St}(\mathbb{Z})]$ be the unique $h$-equivariant map which sends $1$ to $1$. Let $h_1: \mathbb{Z}[W(\pm 1)] \xrightarrow{\chi_\mu} \mathbb{Z}[\text{St}(\mathbb{Z})] \xrightarrow{\chi_{\text{St}}} \mathbb{Z}$ be the $h$-equivariant map given by $h_1([w_{ij}(1)]) = \{a_{ij}^1[e_{ij}^{-1}] + e_{ij}^{-1}e_{ij}^1[a_{ij}^1]\}$. Then it is clear that $3h_1 = h_0 \circ h_\chi$.

Let $h_2: \mathbb{Z}[W(\pm 1)] \xrightarrow{f_W} \mathbb{Z}[\text{St}(\mathbb{Z})] \xrightarrow{f_{\text{St}}} \mathbb{Z}$ be defined on $f_W$ by the following equations.

1) If $y = [w_{ij}(1), w_{kl}(1)]$ where $i, j, k, l$ are distinct, let $h_2([y])$

$= (1 + ab^{-1} + cd^{-1} + ab^{-1}cd^{-1})[[a, c]]$

$+ (ab^{-1} + ab^{-1}cd^{-1})[[c, b]]$

$+ (cd^{-1} + ab^{-1}cd^{-1})[[d, a]]$

$+ ab^{-1}cd^{-1}[[d, b]]$

where $a = e_{ij}^1$, $b = e_{ji}^1$, $c = e_{kl}^1$, $d = e_{lk}^1$. This can be better understood
by examining the following partial graph.

From the partial graph it is obvious that \( \partial h_2([y]) = h_1 \partial([y]) \) but this can also be checked algebraically.

2) If \( y = w_{ij}(l) w_{ik}(l) w_{ij}(l)^{-1} w_{jk}(l) \) where \( i, j, k \), are distinct, let \( h_2([y]) \)

\[
\begin{align*}
&= (e^{-1}c^{-1} + a^{-1} + e^{-1}fe^{-1}c^{-1} + e^{-1}fab^{-1})[[a, c]] \\
&+ (e^{-1}c^{-1} + e^{-1}fe^{-1}c^{-1})[[c, e, a]] \\
&+ (ab^{-1}c^{-1} + e^{-1}fab^{-1}c^{-1})[[e, b]] \\
&+ (ab^{-1}c^{-1} + e^{-1}fab^{-1}c^{-1})[[c, e, b]] \\
&+ e^{-1}[[a, f]] \\
&+ ab^{-1}cd^{-1}[[f, b]d^{-1}] \\
&+ ab^{-1}cd^{-1}[[a, f]d^{-1}]
\end{align*}
\]

where \( a = e_{ij}^1 \), \( b = e_{ji}^1 \), \( c = e_{ik}^1 \), \( d = e_{ki}^1 \), \( e = e_{jk}^1 \), \( f = e_{kj}^1 \).
The corresponding partial graph is:

3) If \( y = w_{ij}(1) w_{ki}(1) w_{ij}(l)^{-1} w_{kj}(l) \) where \( i, j, k \) are distinct, then \( h_2([y]) \)

\[
= (f^{-1} + f^{-1} ef^{-1})[[f, a]]
+ (f^{-1} c^{-1} + f^{-1} ab^{-1} c^{-1})[[c, a]]
+ (ab^{-1} f^{-1} + f^{-1} eab^{-1} f^{-1})[[a, d]]
+ (ab^{-1} f^{-1} + f^{-1} eab^{-1} f^{-1})[[d, f]]
+ f^{-1} c^{-1}[[a, e] c^{-1}]
\]
\[ f^{-1}ab^{-1}c^{-1}d^{-1}[[b, c]e^{-1}] + f^{-1}ab^{-1}c^{-1}d^{-1}[[e, b]] \]

where \( a - f \) are the same as in (2).

The corresponding partial graph is

\[ a \rightarrow b \rightarrow c \rightarrow d \]

4) If \( y = w_{ij}(1)w_{jk}(1)w_{ij}(1)^{-1}w_{ik}(1)^{-1} \) where \( i, j, k \) are distinct, then let \( h_{2}(\{y\}) \)

\[ = (1 + cd^{-1})[[a, c]] + (cd^{-1} + cd^{-1}ab^{-1})[[f, a]] + (ab^{-1} + cab^{-1}f^{-1})([a[2, c, b]] + [[c, c]] + [[a, e]c^{-1}]) + cd^{-1}[[d, a]f^{-1}] + cd^{-1}a[[f, d]] + cab^{-1}f^{-1}[[b, d]] + cab^{-1}f^{-1}[d[b, f]] \]

where \( a - f \) are the same as in (2).
5) If \( y = w_{ij}(1) w_{kj}(1) w_{ij}^{-1}(1) w_{ki}(1) \) where \( i, j, k \) are distinct, then let \( h_2([y]) \)
\[
= (1 + ab^{-1} + dc^{-1} + dc^{-1}ab^{-1})[[a, f]] \\
+ (1 + dc^{-1})[f[a, d]] \\
+ (a + dc^{-1}a)[[d, f]] \\
+ (ab^{-1} + dc^{-1}ab^{-1})([[d, b]] + [[f, b]d^{-1}]) \\
+ dc^{-1}[[c, a]] \\
+ dc^{-1}ab^{-1}([[b, c]e^{-1}] + [[e, c]] + [c[e, a]])
\]
where \( a - f \) are the same as in (2).
The corresponding partial graph is

We must now compute $\chi_c h_2$.

**Theorem 6.1:** $\chi_c h_2([y]) = 1$ if $y$ is a relation of type 3 and $\chi_c h_2([y]) = 0$ if $y$ is a relation of type 1, 2, 4, 5.

**Proof:** If $y$ is a type 1 relation for $W(\bar{1} 1)$ then $\chi_c h_2([y]) = 0$ because $h_2([y])$ contains no relevant relation. For the other elements of
the only relevant Steinberg relations that occur are \([a, c], [b, e], [d, f]\), and their inverses.

If \(g \in \text{St}(\mathbb{Z})\) and \((r_{pq}) = (s_{qp})^{-1}\) is the image of \(g\) in \(\text{GL}(\mathbb{Z}_2)\), then \(\chi_c(g[[e_{ij}^1, e_{ik}^1]]) = \sum_p s_{jp} s_{kp} r_{pi}\). Using this formula one can easily calculate \(\chi_c\) for all the relevant terms that occur in all the \(h_2([y])\)'s. The result is that \(\chi_c\) is zero on all the terms except the last term in (3) where it is 1. In fact if \(g = f^{-1}ab^{-1}c^{-1}e^{-1}\) then

\[
(r_{pq}) = \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
i & j & k
\end{array}
\quad (s_{qp}) = \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
k & 0 & 1 \\
i & j & k
\end{array}
\]

and \(\chi_c(g[[e, b]]) = \sum_p s_{ip} s_{kp} r_{pj} = 1\).

**Theorem 6.2:** There exists a graph \(\mathcal{P} \in \mathcal{P}(w(\pm 1))\) with the property that \(\chi_c h_2(\mathcal{P}) = 1\).

**Proof:** All we have to do is to find a graph which contains an odd number of type 3 relations. Here it is.

(See next page.)
The numbers at the vertices indicate the base point direction and the type of relation that occurs at the vertex. Each type of relation occurs an odd number of times.
Corollary 6.3: \( \chi: K_3(\mathbb{Z}) \to \mathbb{Z}_2 \) is surjective and thus \( K_3(\mathbb{Z}) \) has at least 48 elements.

Proof: It is well known (see [Q]) that \( H_3^s \cong H_3 T(1) \to H_3 \text{St}(\mathbb{Z}) \cong K_3(\mathbb{Z}) \) is injective, and by theorem 5.9 its image is contained in the kernel of \( \chi \).

The example given in (6.2) was originally discovered by the author by multiplying together the nontrivial elements of \( K_1(\mathbb{Z}) \) and \( K_2(\mathbb{Z}) \) using Loday's formula (see [L]), by displaying this element as a graph, and be deforming the graph (and adding and subtracting second order Steinberg relations) until it was the sum of 8 equal pieces.

§7. Application to Pseudoisotopy.

Let \( M \) be a compact smooth manifold. A pseudoisotopy of \( M \) modulo \( \mathcal{A}M \) is a self-diffeomorphism of \( M \times I \) which keeps \( M \times \{0\} \cup (\mathcal{A}M) \times I \) pointwise fixed. Let \( \mathcal{P}(M, \mathcal{A}M) \) denote the space of all pseudoisotopies of \( M \) modulo \( \mathcal{A}M \) with the \( C^\infty \)-topology.

Theorem 7.1: If \( \dim M \geq 5 \), \( \Pi_1 M = \Pi \) and \( \Pi_2 M = 0 \), then there is an exact sequence

\[
\begin{array}{c}
\chi W_{\Pi} \\
\Phi \Pi \end{array} \to \begin{array}{c}
W_1(\Pi; \mathbb{Z}_2) \\
\Pi_0 \mathcal{P}(M, \mathcal{A}M) \\
W_2(\Pi) \to 0
\end{array}
\]

where the "Whitehead groups" are defined below.

a) \( W_2(\Pi) \) is the cokernel of the following map induced by inclusion.
\[ H_2 \mathcal{M}(\Pi') \rightarrow H_2 \mathcal{L}(\mathbb{Z}[\Pi])' \cong K_3(\mathbb{Z}[\Pi]). \]

b) \( Wh_1(\Pi; \mathbb{Z}_2) = H_0(\Pi; \mathbb{Z}_2[\Pi])/\mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) represents the image of \( H_0(1; \mathbb{Z}_2[1]) \).

c) \( Wh_3(\Pi) \) is the quotient of \( K_3(\mathbb{Z}[\Pi]) = H_3 \text{St}(\mathbb{Z}[\Pi]) \) by the sum of \( K_3(\mathbb{Z}) \) and the image of \( H_3 T(\Pi) \).

d) \( \chi_{Wh}: Wh_3(\Pi) \rightarrow Wh_1(\Pi; \mathbb{Z}_2) \) is the map \( \chi \) induced by \( \chi: K_3(\mathbb{Z}[\Pi]) \rightarrow H_0(\Pi; \mathbb{Z}_2[\Pi]). \)

We shall assume that the reader is familiar with the Hatcher-Wagoner techniques for studying \( H_0(\mathcal{M}, \mathcal{X}) \). Suppose that we have a generic one-parameter family of functions \( f_t \) on \( M \times I \) with critical points of index \( i \) and \( i+1 \) and with only \( i+1/i+1 \) handle additions. Suppose also that we have a generic gradient-like vector field \( v_t \) for \( f_t \). Then we can associate to the pair \( (f_t, v_t) \) an element of \( Wh_1(\Pi; \mathbb{Z}_2) \) in the following way.

1) Choose a framing for the tangent bundle of the unstable manifold at each Morse point. This framing should vary smoothly with \( t \).

2) At each birth-death point and at each \( i+1/i+1 \) handle addition the framings should be deformed so that they agree up to the sign of the last vector.

3) The framings for the index 1 Morse points give a diffeomorphism of each lower index unstable sphere with the standard sphere \((X)\).
4) Choose a path from each birth point to a base point $*$ of $M \times I \times I$.

5) Number the birth points 1, 2, 3, etc.

6) We can now associate to each $i + 1/i + 1$ handle addition an elementary operation $c_{jk}^u$ where $u \in \Pi$.

7) Let $J$ be the subspace of the set of lower index unstable spheres $n - 1 \cup \coprod_{p} S^{n-i+1} \times I$ given by $i + 1/i$ intersections. ($n = \dim M$) Then $J$ is a framed 1-complex, that is, the generic points of $J$ are equipped with smoothly varying normal framings and these framings agree at singular points except for the last vector.

8) If $x \in J$, then $x$ determines a parameter value $t(x)$. This determines an invertible matrix $(r_{pq}(x)) \in GL(\mathbb{Z}[\Pi])$ which is the product of the elementary operations associated with the handle addition which occur before time $t(x)$. Let $(s_{qp}(x)) = (r_{pq}(x))^{-1}$.

9) If $x \in J$ is a generic point, and if $x$ is an intersection between the $p$-th lower index unstable sphere with the $q$-th upper index stable sphere, then $x$ determines an element $\sigma(x) \in \Pi$ by $\sigma(x) = [\lambda_p^{-1} \lambda(x) \lambda_q]$, i.e. this is the path given by going from $*$ to the $q$-th birth point, following the $q$-th upper index Morse line to time $t(x)$, going down the integral curve determined by $x$ until one reaches the $p$-th lower index Morse point at time $t(x)$, following this Morse line back to the $p$-th birth point and going back to $*$.

Let $c(x) = s_{qp}(x) \in \mathbb{Z}[\Pi]$. Let $\bar{c}(x)$ be the image of $c(x)$.
in \( \mathbb{Z}_2[\Pi] \).

10) If \( u \in \Pi \), let \( J_u \) be the closure in \( J \) of the set of generic points \( x \) such that \( \langle u, \sigma(x) \circ(x) \rangle = u \). Then \( J_u \) is a closed 1-manifold with "corners" unless \( u = 1 \).

11) The corners can be straightened out in a canonical way and we get a closed framed 1-manifold in \( \bigcup_{p} \mathbb{S}^{n-1+1} \times I \) for every nontrivial element of \( \Pi \). This determines an element of \( \mathbb{Z}_2[\Pi]/\mathbb{Z}_2[1] \) by adding up the elements \( u \in \Pi \) for which \( J_u \) is nontrivially framed. \( J_u \) is empty except for a finite number of \( u \in \Pi \) so there are only finitely many such elements.

12) Because of the choices made the element of \( \mathbb{Z}_2[\Pi]/\mathbb{Z}_2[1] \) is not well defined but its image in \( \text{Wh}_1(\Pi; \mathbb{Z}_2) \) is well defined and will be denoted \( k(f'_t, v'_t) \).

Let \( c \) denote the choices made in (1), (2), (4), and (5). Then to the triple \((f'_t, v'_t, c)\) we can associate a sequence of elementary operations \( g(f'_t, v'_t, c) \in F \) where \( F \) is the free group generated by symbols \( e_{ij}^u \), \( i \neq j \), \( u \in \Pi \). Note that the image of \( g(f'_t, v'_t, c) \) in \( \text{GL}(\mathbb{Z}[\Pi]) \) must be a monomial matrix with entries from \( \pm \Pi \).

**Lemma 7.2:** Suppose that \((f'_t, v'_t, c)\) is deformed by commuting two consecutive letters in \( g(f'_t, v'_t, c) \) of the form \( e_{jk}^u, e_{j\ell}^v \). That is, we have a new triple \((f''_t, v''_t, c')\) with \( g(f''_t, v''_t, c') = f[e_{jk}^u, e_{j\ell}^v]g(f'_t, v'_t, c) \). Then \( k(f''_t, v''_t) = k(f'_t, v'_t) + \chi_c(f[e_{jk}^u, e_{j\ell}^v]) \).

**Proof:** The above deformation results in the following deformation of \( J \cap \bigcup_{p} \mathbb{S}^{n-1+1} \times I \).
If the $p \setminus k$ and $p \setminus \ell$ (\setminus = "under") segments both belong to $J_\wedge$, then there are two possibilities for $J_\wedge$.

a) $J_\wedge \cap S^{n-1+1} \times I = \begin{array}{c}
\begin{array}{c}
p \setminus k \\
p \setminus \ell
\end{array}
\end{array}$

b) $J'_\wedge \cap S^{n-1+1} \times I = \begin{array}{c}
\begin{array}{c}
p \setminus k \\
p \setminus \ell
\end{array}
\end{array}$

In both cases the deformation changes the framed bordism class of
\[ J_w \cap S^{n-1} \times I. \] The condition that \( p \) and \( q \) both belong to \( J_w \) is expresses algebraically (see (10) above) by \( \langle w, \sigma(p \setminus k) \tau(p \setminus l) \rangle = w, \) where

\[
\begin{align*}
\tau(p \setminus k) &= s_{kp} \\
\tau(p \setminus l) &= s_{lp} \\
\sigma(p \setminus k) &= \sigma(p \setminus l)u \\
\sigma(p \setminus l) &= \sigma(p \setminus l)v
\end{align*}
\]

and \( \left( r_{pq}^{-1} \right) = \left( s_{qp}^{-1} \right) \) is the image of \( f \) in \( GL(\mathbb{Z}_2[[\Pi]]) \). However the deformation (*) occurs in many places in \( J \cap S^{n-1} \times I \) depending on the number of geometric \( p \setminus q \) intersections which occur. If we add these up we see that the framed bordism class of \( J_w \cap S^{n-1} \times I \) changes if and only if \( \langle w, \sigma_{kp} \rangle = \langle w, \sigma_{lp} \rangle = w \) is true for an odd number of \( \sigma \) in \( r_{pq} \in \mathbb{Z}_2[[\Pi]] \).

This can also be expressed by the formula

\[ \langle w, r_{pq} \sigma_{kp}, \sigma_{lp} \rangle = w. \]

Adding these up for all \( p \) we get that the framed cobordism class of \( J_w \) changes if and only if \( \langle w, \chi_c(f[[e_{jk}^v, e_{jk}^e]]) \rangle = w. \) Thus \( \chi_c(f[[e_{jk}^v, e_{jk}^e]]) \) is the set of all \( w \)'s for which \( J_w \) changes.

**Remark 7.3:** One consequence of (7.2) is that even if \( \chi_{wm} \) is trivial and we get an exact sequence

\[ 0 \longrightarrow Wh_1(\Pi; \mathbb{Z}_2) \longrightarrow \pi_0\mathcal{G}(\mathcal{M}, \mathcal{M}) \longrightarrow Wh_2(\Pi) \longrightarrow 0, \]

the splitting result for this sequence fails because \( \chi_c \) is surjective.
Lemma 7.4: If \((f_t, v_t, c)\) is deformed by changing \(g(f_t, v_t, c)\) by an "irrelevant" Steinberg relation or by cancelling two handle additions (by definition this doesn't change \(g\)) \(k(f_t, v_t)\) remains unchanged.

Proof: If \(g(f_t, v_t, c)\) is changed by a relation \([e_{jk}^u, e_{lm}^v]\) where \(j \neq k, m \neq l\) then nothing happens because \(J\) is changed by an isotopy. If two handle additions are cancelled or created, \(J\) changes by a concordance:

\[
\begin{array}{c}
\text{or} \\
\end{array}
\]

If \(g(f_t, v_t, c)\) is changed by a relation of the form \([e_{jk}^u, e_{kl}^v]e_{jk}^{-uv}\), then \(J \cap S^{n-1+1}_p \times I\) changes as follows:

\[
\begin{array}{c}
\end{array}
\]

If we examine all eight possibilities for \(J_w \cap S^{n-1+1}_p \times I\) we see that nothing happens. To prove Theorem 7.1 it is sufficient to show

Theorem 7.5: If \((f_t, v_t)\) is a lens-shaped one-parameter family of functions on \(M \times I\) with no handle additions then \((f_t, v_t)\) can be deformed to a one-parameter family with no singularities if and only
if \( k(f_t, v_t) \) is in the image of \( \chi_{Wh} \).

**Proof:** Suppose that \( k(f_t, v_t) \) is in the image of \( \chi_{Wh} \). Then there is an element \( P \) of \( \overline{P}(\text{St}(\mathbb{Z}[\Pi])) \) which maps to \( k(f_t, v_t) \). We can construct a thimble-shaped two parameter family of functions on \( M \times I \) whose handle addition pattern is given by \( P \). By Lemmas 7.2, 7.4 this family is a null-deformation of a lens-shaped family with no handle additions with \( Wh_1(\Pi; \mathbb{Z}_2) \) invariant equal to \( k(f_t, v_t) \). It is well known that two such families with the same \( Wh_1(\Pi; \mathbb{Z}_2) \) invariant can be deformed into each other.

The converse is not so easy. Suppose that there is a null-deformation of \( (f_t, v_t) \). This produces a two-parameter family with "boundary" \( (f_t, v_t) \). By a complicated procedure one can deform this two-parameter family fixing the boundary so that it is thimble-shaped in the same two indices as \( f_t \). The exchange points can then be eliminated and we can look at the handle additions and we can read off a graph \( P \) in \( \overline{P}(\text{St}(\mathbb{Z}[\Pi])) \) whose image in \( Wh_1(\Pi; \mathbb{Z}_2) \) is \( k(f_t, v_t) \). The details can be found in [I].
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